A REVIEW ON ACFA

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Abstract. We give a brief summary of Chatzidakis and Hrushovski's [2] result on simplicity of the theory of the existentially closed difference fields and Hrushovski. This includes and introduction to generic structures based on Chatzidakis and Pillay's [3].

1. generic automorphisms

Let X be a topologic space and $A \subseteq X$. A is said to be *nowheredense* if its closure $\bar{A}$ has empty interior. A is said to be *meager* if A is a countable union of nowheredense subsets of X. A is said to be *generic* in X if it is comeager, i.e. if its complement is meager.

This are well known definitions. The new thing is that we can relativize this notions in the following way:

Definition 1.1. Let X be a topologic space and $A \subseteq X$.

(1) A is said to be $\aleph_1$-*meager* if A is an $\aleph_1$ union of closed nowheredense subsets of X.
(2) A is said to be $\aleph_1$-*generic* in X if it is $\aleph_1$-comeager, i.e. if its complement is $\aleph_1$-meager.

Let T be a first order theory and M a saturated model for T. We denote by $Aut(M)$ the group of all automorphisms of M. In the same way, $Aut_A(M)$ denotes the set of all the automorphisms of M that fix A pointwise, where $A \subseteq M$. If $G = Aut(M)$, G is a topological group where the basic open neighborhoods of the identity are the subgrups $Aut_A(M)$ where $A \subseteq M$ is finite.

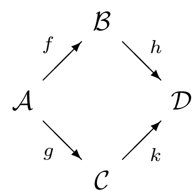
But as in the previous case, this topology can be redefined:

Definition 1.2. We define the $\aleph_1$-topology on $Aut(M)$ where the basic open subsets of $Aut(M)$ are the subgroups $Aut_A(M)$, where $|A| = \aleph_0$.

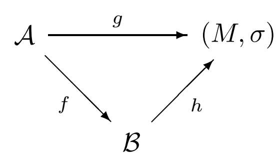
Following [4], let $\mathfrak{C}$ be the monster model of T. A *local automorphism* $\mathcal{A}$ is a couple (A, α), where A is an algebraically closed subset of $\mathcal{C}$ and α is an automorphism of A. A *morphism* of local automorphisms $f : \mathcal{A} \to \mathcal{B}$ is an elementary function $f : A \to B$ such that for every $a \in A$, $f(\alpha(a)) = \beta(f(a))$.

Definition 1.3. T is said to have the *Amalgamation Property for Automorphisms* (PAPA from its frech abreviation) if given three local automorphisms $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ and given two morphisms of local automorphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{A} \to \mathcal{C}$, there exist a local automorphism $\mathcal{D}$ and two morphisms of local automorphisms

$h : \mathcal{B} \to \mathcal{D}$ and $\mathcal{C} \to \mathcal{D}$ such that the following diagram commutes:



Definition 1.4. Let $\sigma \in Aut(M)$. We say that σ is $\aleph_1$-*generic* if the following holds: Given two countable local automorphisms $\mathcal{A}$, $\mathcal{B}$ and given two morphisms of local automorphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{A} \to (M, \sigma)$, there exist a morphism of local automorphisms $h : \mathcal{B} \to (M, \sigma)$ such that the following diagram commutes:



Theorem 1.5 (Lemma 3.11 in [4])**.** *Under CH, if T has the PAPA, the set of $\aleph_1$-generic automorphisms σ of M is $\aleph_1$-generic in $Aut(M)$ with the $\aleph_1$-topology.*

Given a first order theory T, a model complete theory T' is said to be *model companion* of T if every model of T is substructure of a model of T' and viceversa.

Theorem 1.6 (Theorem 8.14 in [7])**.** *T has a model companion if and only if the class $\mathcal{K}$ of the existentially closed models of T is an elementary class and, in that case, the model companion of T is the theory $T' = Th(\mathcal{K})$.*

Let T'_σ be the model companion of $T \cup \{\sigma \text{ es un automorfismo}\}$

Definition 1.7. The interpretations of σ in models of T'_σ, if it exists, are called *generic automorphisms* of T.

Theorem 1.8 (Proposition 3.5 in [3])**.** *If T'_σ exists, the generic automorphisms of T are $\aleph_1$-generic in the monster model of T.*

2. T'_σ for T stable

Theorem 2.1 (Kim-Pillay Theorem, Theorem 1.4 in [3])**.** *Let T be a theory. Supose T has a notion of independence that satisfies the following properties:*

(1) For any $p(x) \in S(A)$ and $B \supseteq A$, there is $q(x) \in B$ such that $q(x) \supseteq p(x)$ and $q(x)$ is independent over A.
(2) $tp(a/B)$ is independent over $A \subseteq B$ if and only if for all tuples b from B, $tp(b/A \cup a)$ is independent over A.
(3) For any $A \subseteq B \subseteq C$ and a, $tp(a(C)$ is independent over A if and only if $tp(a/C)$ is independent over B and $tp(a/B)$ is independent over A.
(4) For any a, and B there is some subset A such that $|A| \leq |T|$ and $tp(a/b)$ is independent over A.

(5) *Let $M \models T$ and A, $B \supseteq M$ such that $tp(A/B)$ is independent over M. Let $p(x) \in S(M)$ and $p_1(x) \in S(A)$, $p_2(x) \in S(B)$ such that p_1, $p_2 \supseteq p$. Then there exists an independent extension $q(x)$ of $p(x)$ which extends $p_1(x) \cup_2 p_2(x)$.*

Then T is simple.

Definition 2.2. Let $(M, \sigma) \models T'_\sigma$, and $A \subseteq M$. We define $acl_\sigma(A)$ to be the algebraic closure of the closure of A under σ and σ^{-1}.

Theorem 2.3 (Lema 3.6 in [3])**.** *$acl_\sigma(A)$ is the algebraic closure of A in (M, σ).*

Definition 2.4. Let A, B and $E \subseteq (M, \sigma)$. A and B are said to be *independent* over E, if $tp_T(acl_\sigma(A)/acl_\sigma(B))$ doesnot fork over $acl(E)$ in the sense of forking in T.

Theorem 2.5 (Theorem 3.7 in [3])**.** *Supose T stable and $(M, \sigma) \models T'_\sigma$. Let $\bar{a}$, $\bar{b}$, $\bar{c}_1$ and $\bar{c}_2$ be independent over M. Then there exists a $\bar{c}$ which is independent from (a, b) over E and realising $tp(\bar{c}_1/acl_\sigma(E, \bar{a})) \cup tp(\bar{c}_2/acl_\sigma(E, \bar{b}))$*

Theorem 2.6 (Corollary 3.8 in [3])**.** *If T is stable and T'_σ exists, then T'_σ is simple.*

Proof. By Theorems 2.5 and 2.1. □

3. Difference Fields

Definition 3.1. A *difference field* is a field K provided with an endomorphism $\sigma : K \to K$.

Definition 3.2. Let ACFA (for *Algebraic Closed Field with an Automorphism*) be the theory axiomatized by sentences expressing the following:

(1) σ is an automorphism of K.
(2) K is algebraically closed
(3) Given varieties U and $V \subseteq U \times \sigma(U)$ projecting generically onto U and $\sigma(U)$, and every algebraic set $W \subsetneq V$, there is $a \in U(K)$ such that $(a, \sigma(a)) \in V \setminus W$

Theorem 3.3 (Theorem 3.2 in [1])**.** *Every difference field embeds in a model of ACFA.*

Theorem 3.4 (Corollary 3.3 in [1])**.** *ACFA is model complete.*

Corolary 3.5. *Let $T = ACF$. Then T'_σ exists and $T'_\sigma = ACFA$.*

Definition 3.6. Let (K, σ) a difference field. By $Fix_K(\sigma)$ we denote the fixed field of σ in K i.e. The field of elements in K fixed pointwise by σ.

Theorem 3.7. *The formula $\phi(x, y) : \exists t \in Fix_K(\sigma)(y - x = t^2)$ in ACFA has the independence property.*

Corolary 3.8. *ACFA is simple not stable.*

Proof. By 2.6. □

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