MODEL THEORY OF HILBERT SPACES EXPANDED WITH AN UNITAL ABELIAN C^*-ALGEBRA OF BOUNDED OPERATORS

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Abstract. We study the model theory of a Hilbert space H, from the point of view of continuous logic, expanded with all the operators that belong to an unital abelian C^*-algebra $\mathcal{A}$ of operators on H. We show that definable operators in this structure are the ones in $\mathcal{A}$. Also, we show for every $v \in \mathcal{H}$ the type $tp(v/\emptyset)$ corresponds to the Radon measure over $Sp(\mathcal{A})$ defined by v, and therefore admits quantifier elimination. We also show that it is $|Sp(\mathcal{A})|$-stable, give an explicit independence relation for it and characterize orthogonality and domination of types.

1. Introduction

This paper deals with a complex Hilbert space H expanded by a an abelian C^*-algebra of bounded operators from the point of view of continuous logic (see [7] and [6]). This is inspired in [3] and [9] where the case of a Hilbert space expansion by some kinds of operators were studied. We start with an unital abelian C^*-algebra of operators over a Hilbert space H. According to [6], we study the model theory of H as a metric structure of only one sort:

$$\mathcal{H} = (H_1, 0, -, i, \tfrac{x+y}{2}, \|\cdot\|, (S)_{S\in\mathcal{A}_1})$$

where H_1 is the unit ball in H; 0 is the zero vector in H; $- : H_1 \to H_1$ is the function that to any vector $v \in H_1$ assigns the vector $-v$; $i : H_1 \to H_1$ is the function that to any vector $v \in H_1$ assigns the vector iv where $i^2 = -1$; $\frac{x+y}{2} : H_1 \times H_1 \to H_1$ is the function that to any couple of vectors v, $w \in H_1$ assigns the the vector $\frac{v+w}{2}$; $\|\cdot\| : H_1 \to [0,1]$ is the norm function; $\mathcal{A}$ is an abelian C^*-algebra of bounded operators on H, and $\mathcal{A}_1$ is the unit ball in $\mathcal{A}$. The metric is given by $d(v,w) = \|\frac{v-w}{2}\|$. Briefly, $\mathcal{H}$ will be refered to as $(H, \mathcal{A})$.

It is worthy notting that with the previous language, we can define inner product taking into account that for every v, $w \in H_1$, $\langle v \mid w\rangle = \|\frac{v+w}{2}\|^2 - \|\frac{v-w}{2}\|^2 + i(\|\frac{v+iw}{2}\|^2 - \|\frac{v-iw}{2}\|^2)$. Because of this reason, we will make free mention to inner product as if that were included in the language. In most arguments, we will forget this formal point of view, and will treat H directly, seemingly as we do in first order structures, as well. To know more about continuous logic point of view of Banach spaces, which can be extended easily to Hilbert spaces, please see [6], Section 2.

From now on, $\tilde{\mathcal{H}}$ will denote an elementary superstructure of $\mathcal{H}$ which is saturated and homogeneous, i.e. a monster model.

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The main results in this paper are the following:

Theorem 1.1. *The definable operators over (H, A) are exactly the ones in $\mathcal{A}$. As a result of that the C^*-algebra $\mathcal{A}$ is the greatest set of bounded operators that commute with all the unitary operators that commute with all the elements in $\mathcal{A}$.*

Theorem 1.2. *Let $v, w \in \mathcal{H}$. Then $tp(v/\emptyset) = tp(w/\emptyset)$ if and only if $\mu_v = \mu_w$, where μ_v and μ_w are the measures over $Sp(\mathcal{A})$ corresponding to the vectors v and w.*

Theorem 1.3. *Let $A \subseteq H$ be such that $A = acl(A)$. Let $p, q \in S_1(A)$, let $v \models p$ and $w \models q$. Then, $p \perp_A q$ if and only if $\mu_{P^{\perp}_{acl(A)}(v)} \perp \mu_{P^{\perp}_{acl(A)}(w)}$*

Theorem 1.4. *Let A, $\mathfrak{B}$ be small subsets of $\tilde{H}$ and $p \in S_1(A)$ and $q \in S_1(B)$ two stationary types. Then $p \rhd_C q$ if and only if there exist v $w \in \tilde{\mathcal{H}}$ such that $tp(v/C)$ is a non-forking extension of p, $tp(w/C)$ is a non-forking extension of q and $\mu_{P^{\perp}_{acl(A)} w} \ll \mu_{P^{\perp}_{acl(A)} v}$.*

Theorem 1.5. *Let $T_{\mathcal{A}}$ the theory of Hilbert spaces together with the following conditions:*

(1) For $v \in H_1$ and $S, T \in \mathcal{A}_1$:
$$(ST)v = S(Tv)$$
(2) For $v \in H_1$ and $S, T \in \mathcal{A}_1$:
$$\tfrac{S+T}{2}(v) = \tfrac{Sv+Tv}{2}$$
(3) For $v \in H_1$ and $S \in \mathcal{A}_1$:
$$\langle Sv \mid w\rangle = \langle v \mid S^* w\rangle$$
(4) For $v \in H_1$ and $S \in \mathcal{A}_1$:
$$\|Sv\| \leq \ |S\|\|v\|$$
(5) For $v \in H_1$ and Id the identity operator in $\mathcal{A}$:
$$(iId)v = iv$$
(6) For v, $w \in H_1$, and $S \in \mathcal{A}_1$:
$$S\left(\tfrac{v+w}{2}\right) = \tfrac{Sv+Sw}{2}$$
(7) For $S \in \mathcal{A}_e \cap \mathcal{A}_1$, and every $n \in \mathbb{N}$:
$$\exists u_1 u_2 \cdots u_n \forall i (|\|Su_i\| = 1) \wedge \forall i \neq j (\langle Su_i | Su_j\rangle = 0)$$
(8) Let $S \in \mathcal{A}_d \cap \mathcal{A}_1$, $rank(S) = n \in \mathbb{N}$,
$$\exists u_1 u_2 \cdots u_n \forall i (|\|Su_i\| = 1) \wedge \forall i \neq j (\langle Su_i | Su_j\rangle = 0) \wedge \forall v (Sv = \sum_{k=1}^{m} \langle Sv | Su_i\rangle u_i) = 0$$

Then $T_{\mathcal{A}}$ axiomatizes the theory of $\mathcal{H}$, admits quantifier elimination, and is κ-stable for $\kappa \geq |Sp(\mathcal{A})|$

The first author and Berenstein ([3]) studied the theory of the structure $(H, +, 0, \langle | \rangle, U)$ where U is a unitary operator in the case when the spectrum is countable. As it was said before, most results in this paper are generalizations of results there. Previous to that, Henson and Iovino in [15], observed that this theory is stable. A geometric characterization of forking in such structures was first done by Berenstein and Buechler [10]. The second author, Usvyatsov and Zadka characterized the unitary operators corresponding to generic automorphisms of a Hilbert space as those unitary transformations whose spectrum is S^1 and gave the key ideas used in this paper to characterize domination and orthogonality of types.

In this work we study the definability of bounded operators in $(H, \mathcal{A})$; the relation of types with the operator algebra theory of $\mathcal{A}$; we axiomatize the theory $Th(\mathcal{H})$ and give a specific freeness relation for it, and characterize orthogonality and domination of types. The framework for this work is continuous logic and we assume that the reader is familiar with notions such as definability, definable and algebraic closure, and forking. The background can be found in [7, 8].

This paper is divided as follows: In the section 2, we give a summarized introduction to C^*-algebra theory. In section 3 we study the definability of operators on $\mathcal{H}$, characterize the types over a set as measures on $Sp(\mathcal{A})$ or positive linear functional on $\mathcal{A}$. In section 4 we give an axiomatization of $Th(\mathcal{H})$. In section 5, we characterize algebraic types. Finally, in section 6 we study the stability of the theory $Th(\mathcal{H})$, give an independence relation ammong types, characterize orthogonality and domination between types.

It remains to eventual works, to understand what kind of elimination of imaginaries holds in $\mathcal{H}$ and to study the case of a general C^*-algebra not necessarily abelian or even other operator algebras.

2. Preliminaries: C^*-algebras

In this section we study the basic theory of C^*-algebras and their representations along with some important theorems of measure theory. We define positive linear functionals, states, representation and cyclic representations which are key notion for the work in next sections. The main sources of this section are [18] for elementary operator theory, [19] for measure theory and elementary analysi, [2] for spectral theory of operators and [12] and [17] for C^*-algebra theory.

2.1. **C^*-algebras.** In this subsection we give some elementary facts about C^*-algebras. The main goal of this subsection is Theorem 2.31 which states that Gelfand transform is a *-algebra isomorphism. This will be used in next subsection to relate radon measures (regular loccally finite) measures with positive bounded linear functionals over $\mathcal{A}$ which will be defined in Subsection 2.4. References used here are [18], [12] and [17].

Definition 2.1. Let $\mathcal{A}$ be a complex Banach algebra. $\mathcal{A}$ is called a C^*-algebra if there exists a map $^* : \mathcal{A} \to \mathcal{A}$, called *involution* such that for all a, $b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$:

(1) $(a + b)^* = a^* + b^*$
(2) $(ab)^* = b^* a^*$
(3) $(\alpha a)^* = \bar{\alpha} a^*$
(4) $(a^*)^* = a$
(5) $|a^* a| = |a|^2$

Fact 2.2. $\mathbb{C}$ is a C^*-algebra under complex conjugation.

Definition 2.3.

- Let S be a linear operator from H into H. The operator S is called *bounded* if the set $\{\|Su\| : u \in H, \|u\| = 1\}$ is bounded in $\mathbb{C}$. If A is bounded we define the *norm* of A by:
$$\|S\| = \sup_{u \in H, \|u\|=1} \|Su\|$$
Let H be a Hilbert space. We denote by $B(H)$ the algebra of all bounded linear operators from H to H.
- Given a linear operator $S : H \to H$, its *adjoint operator*, denoted S^* is the unique linear operator $S^* : H \to H$ such that for every $u, v \in H$, $\langle Su|v\rangle = \langle u|S^*v\rangle$.

Remark 2.4. The unicity of the adjoint comes from a duality relation between H and H'. See [18], Volume 1, Chapter VI, Section 2.

Fact 2.5. $B(H)$ is a C^*-algebra under the adjoint operation.

Definition 2.6. Let X be a locally compact Hausdorff space. We set:

$$\begin{aligned} \mathcal{C}_0(X,\mathbb{C}) = \{f \in \mathcal{C}(X,\mathbb{C}) \mid & \text{ for all } \epsilon > 0 \\ & \text{there exist } K \subseteq X \text{ compact such that } |f(x)| < \epsilon \text{ for all } x \in X \setminus K\}. \end{aligned}$$

Fact 2.7. If X is a topological space (resp. locally compact Hausdorff space), $\mathcal{C}(X,\mathbb{C})$ (resp. $\mathcal{C}_0(X,\mathbb{C})$) is a C^*-algebra under the adjoint operation on complex functions.

Definition 2.8. Let $\mathcal{A}$ be a complex Banach algebra with identity. Let $a \in \mathcal{A}$. Then the *spectrum* of a is the set $\sigma_A(a)$ of complex values $\lambda \in \mathbb{C}$ such that the element $a - \lambda 1$ is not invertible. The set $\mathbb{C} \setminus \sigma_A(a)$ is called the *resolvent set* for a and is denoted $\rho_A(a)$. Let S a bounded linear operator on a Hilbert space H. Then $\sigma(S)$ denotes the *spectrum* of S as an element of the algebra $B(H)$ i.e. the set of complex values t such that $S - tI$ is not an invertible operator in $B(H)$; also $\rho(a)$ is the resolvent set of S as an element of the algebra $B(H)$ i.e. the set of complex values t such that $S - tI$ is an invertible operator in $B(H)$.

Definition 2.9. Let $a \in \mathcal{A}$. Then, a is called *selfadjoint* if $a = a^*$. If in addition $\mathcal{A}$ has an identity and $\sigma_A(a) \subseteq [0, \infty)$, then a is called *positive*. The set of positive elements in $\mathcal{A}$ is denoted by $\mathcal{A}_+$.

Fact 2.10. A self-adjoint operator S is positive if and only if $\langle Su|u\rangle \geq 0$ for all $u \in H$.

Notation 2.11. Let a, $b \in \mathcal{A}$. By $a \leq b$ we mean that $b - a$ is positive.

Definition 2.12. A self-adjoint element $a \in \mathcal{A}$ a *square root* of a positive self-adjoint operator $b \in \mathcal{A}$ if $a^*a = a^2 = b$.

Fact 2.13 (Theorem VI.9 in [18] Vol 1)**.** Every positive operator on H admits a unique positive square root.

Definition 2.14.

- Let $\mathcal{A}$ be a C^*-algebra. A *unit* in $\mathcal{A}$ is an element $e \in \mathcal{A}$ such that $ea = ae = a$ for all $a \in \mathcal{A}$.
- A C^*-algebra with a unit is called *unital*.

- Let $\mathcal{A}$ be a C^*-algebra. An *approximate unit* in $\mathcal{A}$ is a net $(a_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{A}$ such that $\lambda < \mu$ implies that $a_\lambda \leq a_\mu$ and $\lim \|x - xa_\lambda\| = 0$ for all $x \in \mathcal{A}$.

Fact 2.15 (Theorem 1.4.2 in [17])**.** Each C^*-algebra contains an approximate unit.

Definition 2.16. Let $\mathcal{A}$ be a *-algebra with a unity. An element $a \in \mathcal{A}$ is called *normal* if $aa^* = a^*a$.

Notation 2.17. Let N be a normal operator on a Hilbert space H. Then C^*-subalgebra of $B(H)$ generated by N and I is denoted by $C^*(N)$.

Definition 2.18. Let $\mathcal{A}$ an unital C^*-algebra and let $a \in \mathcal{A}$ an invertible element. a is called *unitary* if $a^{-1} = a^*$.

Fact 2.19. Let S be a linear operator on H. Then, S is unitary if and only if S is an isometry.

Definition 2.20. Let S a bounded linear operator from H to H. Then,

- S is called of *finite rank* if $\dim(SH) < \infty$.
- S is called *compact* if the image under S of the unit ball is relatively compact.

Fact 2.21 (Theorem VI.13 in [18])**.** A bounded linear operator K is compact if and only if there exists a sequence of finite rank operators F_n such that F_n converges to K in the norm topology.

Notation 2.22. The set of all compact operators in H is denoted by $\mathcal{K}$.

Definition 2.23.

- Let $\mathcal{A}$ be a complex Banach algebra. A subalgebra $\mathcal{I} \subseteq \mathcal{A}$ is called an *ideal* of $\mathcal{A}$, if for every $a \in \mathcal{A}$ and $i \in \mathcal{I}$, ia and ai are in I.
- Let $\mathcal{I}$ be an ideal in $\mathcal{A}$. We define an equivalence relation in $\mathcal{A}$ in the following way:

$$a \simeq_{\mathcal{I}} b \text{ if and only if } b - a \in \mathcal{I}$$

- Let $\mathcal{I}$ be an ideal in $\mathcal{A}$. The *quotient algebra* $\mathcal{A}/\mathcal{I}$ is the set of equivalence classes according $\simeq_{\mathcal{I}}$.

Fact 2.24. Let $\mathcal{I}$ be an ideal in a C^*-algebra $\mathcal{A}$. Then $\mathcal{A}/\mathcal{I}$ is a C^*-algebra.

Fact 2.25. The set $\mathcal{K}(H)$ of all the compact operators on H defines a closed ideal in $B(H)$

Definition 2.26. The algebra $B(H)/\mathcal{K}(H)$ is called the *Calkin algebra* on H. If N is a normal operator on H, the C^*-subalgebra of $B(H)$ generated by equivalence class of N and the equivalence class of I is denoted $\tilde{C}^*(N)$.

From now on, $\mathcal{A}$ will denote a C^*-algebra

Definition 2.27. The set:

$$Sp(\mathcal{A}) = \{\omega : \mathcal{A} \to \mathbb{C} \,|\, \omega \text{ is a } C^*\text{-algebra homomorphism}\},$$

is called the *spectrum* or the *maximal ideal space* of $\mathcal{A}$. This set is considered to have the weak* topology inherited from $\mathcal{A}'$

Theorem 2.28 (Corollary I.2.6 in [12])**.** *If $\mathcal{A}$ is abelian, then $Sp(\mathcal{A})$ is a loccally compact Hausdorff space. If in addition $\mathcal{A}$ has a unit, $Sp(\mathcal{A})$ is compact.*

Corolary 2.29. *Let $\mathcal{A} = C_0(K)$ where K is a locally compact Hausdorff space. Then $Sp(\mathcal{A}) = K$*

Definition 2.30. Let $\mathcal{A}$ be an abelian C^*-algebra. Then the morphism Γ from $\mathcal{A}$ to $\mathcal{C}_0(Sp(\mathcal{A}), \mathbb{C})$ given by:

$$\Gamma(a) : Sp(\mathcal{A}) \to \mathbb{C}$$
$$f \to \Gamma(a)(f) = f(a)$$

for every $a \in \mathcal{A}$ is called the *Gelfand transform*.

Theorem 2.31 (Theorem I.3.1. in [12])**.** *Let $\mathcal{A}$ be an abelian C^*-algebra. Then the Gelfand transform is an isometric *-isomorphism.*

Theorem 2.32 (Proposition 1.1.8. and Remark 1.1.4 in [17])**.** *Let $\mathcal{A}$ be abelian C^*-algebra with unity. Let $a \in \mathcal{A}$ be a normal element, i.e. $aa^* = a^*a$. Let $\mathcal{B}$ be the smallest C^*-subalgebra of $\mathcal{A}$ containing a and 1. Then B is isometrically *-isomorphic to $\mathcal{C}_0(\sigma_A(a), \mathbb{C})$ and $\sigma_A(a) = \sigma_B(a)$.*

2.2. **Functional calculus for bounded operators on H.** In this subsection we study the functional calculus for a bounded normal operator on H. This is important to prove Schur's Lemma 2.44. Reference used here is [2]

Definition 2.33. Let $N \in B(H)$ be a normal operator on H. Let $\hat{}$ be the function that for a polynomial $p(x) = \sum_{i,j=0}^{n} c_{ij} z^i \bar{z}^j$ asigns $\hat{p}(N) = \sum_{i,j=0}^{n} c_{ij} N^i (N^*)^j$

Fact 2.34 (Theorem 2.3.1 in [2])**.** Let $N \in B(H)$ be a normal operator on H. Then the previous function extends uniquely to an isometric C^*-algebra isomorphism of $\mathcal{C}(\sigma(N), \mathbb{C})$ onto $C^*(N)$.

Fact 2.35 (Theorem 2.6.3 in [2])**.** Let X be a compact metrizable space. Let $\pi \in rep(\mathcal{C}(X, \mathbb{C}), H)$ be such that H is the closed linear span of $\pi(A)H$. Then, π extends uniquely to a representation $\tilde{\pi} \in rep(\mathcal{B}(X, \mathbb{C}), H)$.

Corolary 2.36. *Let N be a normal operator on H, let $\mathcal{B}(\sigma(N))$ the algebra of borel functions from $\sigma(N)$ into the complex numbers, and let $B(H)$ the algebra of linear operators on H. Then there exist a isometric monomorphism $\pi : \mathcal{B}(\sigma(N)) \to B(H)$ such that $\pi(\bar{f}) = (\pi(f)^*)$, $\pi(1) = I$ and if $f = \sum_i \sum_j a_{ij} z^i \bar{z}^j$, then $\pi(f) = \sum_i \sum_j a_{ij} N^i (N^*)^j$, where by 1 we denote the constant function on $\sigma(N)$ with value 1.*

Definition 2.37. Given a normal operator A, a complex number λ is called an *eigenvalue* or *punctual spectral value* of A if the operator $A - \lambda I$ is not one to one. The set of punctual spectral value is called *point spectrum* and denoted $\sigma_p(T)$. A complex number λ is called a *continuous spectral value* if the operator $A - \lambda I$ is one to one and the operator $(A - \lambda I)^{-1}$ is densely defined but is unbounded. The set of continuous spaectral values is called *continuous spectrum* and is denoted $\sigma_c(T)$.

Definition 2.38. For $\lambda \in \mathbb{C}$ and $\epsilon > 0$, $\bar{\mathcal{D}}(\lambda, \epsilon)$ denotes de closed disc with center λ and radius ϵ.

Definition 2.39. For any $A \subseteq \mathbb{C}$ we denote the characteristic function of A by $\chi(A)$.

Next Lemma and Lemma 2.72 are product of discussion with Mr Thierry Fack:

Lemma 2.40. *Let N be a normal operator on H. For any $\lambda \in \mathbb{C}$, the following conditions are equivalent:*

i $\lambda \in \sigma(N)$

ii For all $\epsilon > 0$, $\chi_{\bar{D}(\lambda,\epsilon)}(N) \neq 0$

Proof. (i)⇒(ii) Asume that there exists $\epsilon > 0$ such that $\chi_{\bar{D}(\lambda,\epsilon)}(N) = 0$. Let

$$f(z) = \frac{1 - \chi_{\bar{D}(\lambda,\epsilon)}(z)}{z - \lambda}.$$

By Corollary 2.36 (functional calculus), we have that,

$$f(N)(N - \lambda I) = (N - \lambda I)f(N) = I - \chi_{\bar{D}(\lambda,\epsilon)}(N) = I,$$

Because $\chi_{\bar{D}(\lambda,\epsilon)}(N) = 0$. But this means that $N - \lambda I$ is invertible in $B(H)$ and therefore $\lambda \notin \sigma(N)$

(ii)⇒(i) Suppose that $\lambda \notin \sigma(N)$. Then $N - \lambda I$ is invertible in $B(H)$. Let T be an inverse for $N - \lambda I$, then

$$T(N - \lambda I) = I. \tag{1}$$

By Hypótesis, for all $\epsilon > 0$, $\chi_{\bar{D}(\lambda,\epsilon)}(N) \neq 0$. So, for every $\epsilon > 0$ there exists $v_\epsilon \in \chi_{\bar{D}(\lambda,\epsilon)}(N)$ such that $\|v_n\| = 1$ By functional calculus (Fact 2.36),

$$\|N - \lambda I\|^2 = \langle (N - \lambda I)^*(N - \lambda I)\chi_{\bar{D}(\lambda,\epsilon)}(N)(v_\epsilon)|v_\epsilon\rangle = \int_{\bar{D}(\lambda,\epsilon)} |z - \lambda|^2 d(E(\lambda)) \leq \epsilon^2,$$

and hence $N(v_\epsilon) - \lambda v_\epsilon \to 0$ when $\epsilon \to 0$. From (1) we get:

$$v_\epsilon = T(Nv_\epsilon - \lambda v_\epsilon) \to 0 \text{ when } \epsilon \to 0.$$

But on the other hand $\|v_\epsilon\| = 1$ for all $\epsilon > 0$, which is a contradiction.

□

Fact 2.41 (Theorem 2.8.2 in [2])**.** If K is normal compact operator, Then $|\sigma(K)| \leq \aleph_0$ and consists of punctual spectral values with finite dimensional eigenvalues and perhaps one accumulation point in $\lambda = 0$.

Definition 2.42. The *essential spectrum* $\sigma_e(T)$ of a linear operator T is the spectrum of the coset $\tilde{T}$ of T in $\mathcal{C}(H)$.

Lemma 2.43 (Generalized Weyl's criterion for essential spectrum)**.** *Let N be a normal operator. Then, for every $\lambda \in \mathbb{C}$, the following conditions are equivalent:*

i $\lambda \in \sigma_e(N)$

ii For all open neighboorhood V of λ, $\dim(\chi_V(N)H) = \infty$

Proof. (i)⇒(ii) Asume that there exists $\epsilon > 0$ such that $\chi_{\bar{D}(\lambda,\epsilon)}(N)$ finite dimensional. Let

$$f(z) = \frac{1 - \chi_{\bar{D}(\lambda,\epsilon)}(z)}{z - \lambda}.$$

By Fact 2.36 (functional calculus), we have that,

$$f(N)(N - \lambda I) = (N - \lambda I)f(N) = I - \chi_{\bar{D}(\lambda,\epsilon)}(N),$$

Since $\chi_{\bar{D}(\lambda,\epsilon)}(N)$ is finite dimensional, this means that $N - \lambda I$ is invertible modulo compact operators, and therefore $\lambda \notin \sigma_e(N)$

(ii)⇐(i) Suppose that $\lambda \notin \sigma(N)$. Then $N - \lambda I$ is invertible modulo compact operators. Let T an inverse for $N - \lambda I$ in $B(H)/\mathcal{K}(H)$, then

$$T(N - \lambda I) = I + K. \tag{2}$$

By Hypótesis, for all $\epsilon > 0$, $\chi_{\bar{D}(\lambda,\epsilon)}(N)$ is infinite dimensional and contains $ker(N - \lambda i)$ which is finite dimensional since $\lambda \notin \sigma_e(N)$. So, for every $\epsilon > 0$ there exists $v_\epsilon \in \chi_{\bar{D}(\lambda,\epsilon)}(N)$ such that $\|v_n\| = 1$ and $d(v_\epsilon, ker(N - \lambda I)) = 1$ By functional calculus (Fact 2.36),

$$\|N - \lambda I\|^2 = \langle (N - \lambda I)^*(N - \lambda I)\chi_{\bar{D}(\lambda,\epsilon)}(N)(v_\epsilon)|v_\epsilon\rangle = \int_{\bar{D}(\lambda,\epsilon)} |z - \lambda|^2 d(E(\lambda)) \leq \epsilon^2,$$

and hence $N(v_\epsilon) - \lambda v_\epsilon \to 0$ when $\epsilon \to 0$. From (2) we get:

$$v_\epsilon + kv_\epsilon = T(Nv_\epsilon - \lambda v_\epsilon) \to 0 \text{ when } \epsilon \to 0.$$

By compactness of k, there exists a sequence $(v_n) \subseteq (v_\epsilon)$ such that $kv_n \to v$ when $n \to \infty$ for some $v \in H$. It follows that $v_n \to -v$ and, since $\|v_n\| = 1$, we get $\|v\| = 1$. Since $N(v_n) - \lambda v_n \to 0$ when $n \to \infty$, we get $Nv = \lambda v$, and hence:

$$\|v_n - v\| \geq d(v_n, ker(N - \lambda I)) = 1,$$

which is a contradiction.

□

Lemma 2.44 (Schur's Lemma)**.** *Let T be a bounded normal operator on $\mathcal{H}$ such that T commutes with every automorphism of $\mathcal{H}$. There exist $\lambda \in \mathcal{C}$ such that $T = \lambda Id$, where Id is the identity operator in $\mathcal{H}$*

Proof. Let G the group of all unitary operators on $\mathcal{H}$ and let T a normal operator that commutes with every element in G. Let $\lambda \in \sigma(T)$.

Case 1, $\lambda \in \sigma_p(T)$**:** In that case $T - \lambda I$ is not injective so, $\mathcal{H}_\lambda \neq \emptyset$. But $\mathcal{H}_\lambda$ is a A-submodule of $\mathcal{H}$ and by Theorem 2.62, $\mathcal{H}_\lambda = \mathcal{H}$. This implies that $T - \lambda I \equiv 0$ on $\mathcal{H}$, so $T = \lambda I$.

Case 2, $\lambda \in \sigma_c(T)$**:** Let $\epsilon > 0$. By Lemma 2.40 $P_{\bar{D}(\lambda,\epsilon)}\mathcal{H} \neq 0$. But $P_{\bar{D}(\lambda,\epsilon)}\mathcal{H}$ is an G-submodule of $\mathcal{H}$, so by irreducibility of $\mathcal{H}$, $P_{\bar{D}(\lambda,\epsilon)}\mathcal{H} = \mathcal{H}$. Then $\mathcal{H}_\lambda = \cap_\epsilon P_{\bar{D}(\lambda,\epsilon)}\mathcal{H} = \mathcal{H}$, so $T = \lambda I$ on $\mathcal{H}$.

Case 3, $\lambda \in \sigma_r(T)$**:** In this case $\overline{Im(T - \lambda I)} \neq \mathcal{H}$. But $\overline{Im(T - \lambda I)} \neq \mathcal{H}$ is a G-submodule of $\mathcal{H}$. By Theorem 2.62, $\mathcal{H}$ is irreducible and $\overline{Im(T - \lambda I)} = 0$, so $T = \lambda I$ on $\mathcal{H}$.

□

2.3. **Bounded positive linear functionals and Radon measures.** This a very important subsection of preliminaries. The main goal is to understand positive linear functionals over $\mathcal{A}$ and relate them with Radon measures over $Sp(\mathcal{A})$. That is exactly Theorem 2.54. It is important to mention that this measures will correspond to types of vectors in $\mathcal{H}$, and the mesure theoretic relations between then will correspond to model theoretic relations between the corresponding types. References used here are [17] and [19].

Definition 2.45. Let $\mathcal{A}'$ be the dual space of $\mathcal{A}$. An element $\phi \in \mathcal{A}'$ is called *positive* if $\phi(a) \geq 0$ whenever $a \in \mathcal{A}$ is positive. Set of positive functionals is denoted by $\mathcal{A}'_+$.

Fact 2.46. Let $\mathcal{A}$ be an abelian C^*-algebra of operators on a Hilbert space H, and let $v \in \mathcal{H}$. Then the function ϕ_v on $\mathcal{A}$ such that for every $S \in \mathcal{A}$, $\phi_v(T) = \langle Sv \mid v \rangle$ is a positive linear functional.

Proof. Linearity is clear. Let S be a positive selfadjoint operator in $\mathcal{A}$, let Q be its square root and let $v \in H$. Then $\langle Sv \mid v \rangle = \langle Q^*Qv \mid v \rangle = \langle Qv \mid Qv \rangle \geq 0$ □

Remark 2.47. It is easy to proof that $\mathcal{A}'$ is a C^*-algebra: if $\phi \in \mathcal{A}'$ we can define ϕ^* by $\phi^*(a) = \overline{\phi(a^*)}$ for all $a \in \mathcal{A}$.

Definition 2.48. A positive linear functional ϕ on $\mathcal{A}$ is called a *state* if $\|\phi\| = 1$. The set of the states on $\mathcal{A}$ is denoted by $S(\mathcal{A})$ or if there is no confusion, simply S.

Definition 2.49. Let ϕ and ψ positive linear functionals on $\mathcal{A}$. They are called *orthogonal* ($\phi \perp \psi$) if $\|\phi - \psi\| = \|\phi\| + \|\psi\|$. Also, ϕ is called *dominated* by ψ ($\phi \leq \psi$) if the functional $\psi - \phi$ is positive.

Recall the following definitions:

Definition 2.50. Let X be a topological space. Let μ and ν be two complex measures on the borel sets of X. μ is said to be *absolutely continuous* ($\mu \ll \nu$) with respect to ν if for every borel set $V \subseteq X$, $\nu(V) = 0$ implies $\mu(V) = 0$. μ and ν are said *mutually singular* ($\mu \perp \nu$) if $\operatorname{supp}(\mu) \cap \operatorname{supp}(\nu) = \emptyset$.

Theorem 2.51 (Lebesgue decomposition theorem. Theorem 6.10 a) in [19])**.** *Let μ and ν be two borel positive measures on a topological space X. Then there exists borel positive measures ν_1 and ν_2 such that $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$.*

Theorem 2.52 (Radon-Nikodim theorem. Theorem 6.10 6) in[19])**.** *Let μ and ν be two borel positive measures on a topological space X such that $\nu \ll \mu$. Then there exists $f \in L^1(X, \mu)$ such that, for every Borel set V, $\nu(V) = \int_V f d\mu$.*

Theorem 2.53 (Riesz representation theorem. Theorem 6.19 in [19])**.** *Let X be a locally compact Hausdorff space. Then any bounded linear functional Φ on $\mathcal{C}_0(X, \mathbb{C})$ is represented by a single complex borel regular measure in the sense that:*

$$\Phi f = \int_X f d\mu,$$

for every $f \in \mathcal{C}_0(X, \mathbb{C})$. Moreover, $\|\phi\| = |\mu|(X)$.

Theorem 2.54. *Let X be a locally compact Hausdorff space. The Radon (locally finite and inner regular) measures on X are in correspondence with the positive linear functionals on $\mathcal{C}_0(X, \mathbb{C})$ and this correspondence is isometric.*

Proof. Let μ be a positive Radon measure on X. Let ϕ_μ the functional on $\mathcal{C}_0(X, \mathbb{C})$ given by $\phi_\mu(f) = \int_X f d\mu$. Then ϕ_μ is positive. Conversely, By the Riesz representation theorem any positive linear functional ϕ on $\mathcal{C}_0(X, \mathbb{C})$ determines a positive measure μ_ϕ. The Riesz representation theorem guarantees that the correspondence is isometric too. □

Lemma 2.55. *$\mu \ll \nu$ if and only if there is a positive real number M such that $\phi_\mu \leq M\phi_\nu$.*

Proof. Let $\phi \leq \psi$ and V such that $\mu_\psi(V) = O$. Then we have that $\mu_\psi = \mu_\phi + \mu_{\psi-\phi}$. Since both μ_ϕ and $\mu_{\psi-\phi}$ are positive measures, $0 = \mu_{\psi-\phi}(V) = \mu_\phi(V)$ and $\mu_\phi \ll \mu_\psi$. Assume now that $\mu \ll \nu$. By Radon-Nikodim theorem, there exists $f \in$

$L^1(X,\nu)$ positive such that $d\mu = f d\nu$. Let $M = \int_X f d\nu$ then $d(\nu-\mu) = (M-f)d\nu$ defines a positive Radon measure corresponding to the functional $M\psi - \phi$. □

Lemma 2.56. *$\mu \perp \nu$ if and only $\phi_m \perp \phi_\nu$.*

Proof. Suppose $\mu \perp \nu$. Then $\|\phi_\mu - \phi_\nu\| = |\mu-\nu|(X) = |\mu(X)| + |\nu(X)| = \|\phi_\mu\| + \|\phi_\nu\|$. Conversely, suppose that $\phi \perp \psi$. Take $A \subseteq \mathrm{supp}(\phi)$ and $B \subseteq \mathrm{supp}(\psi)$. Then $\mu_\phi(B) = \phi(\chi_B) = 0$ and $\mu_\psi(A) = \psi(\chi_A) = 0$. □

From now on, let $\mathcal{A}$ be an abelian C^*-algebra of operators in a Hilbert space H, and let $\mathcal{M}$ be its strong closure in $B(H)$

Theorem 2.57. *Given $v \in H$, the positive linear functional ϕ_v defined in Fact 2.46 defines Radon measure over $Sp(\mathcal{A})$. This measure will be called the spectral measure defined by v and will be denoted μ_v*

Fact 2.58 (Generalized Luzin's theorem, Theorem 2.7.3 in [17])**.** For each $b \in \mathcal{M}$, each projection $p_0 \in \mathcal{M}$ and each set $\{v_1, \dots, v_n\} \subseteq H$, there is a projection $p \in \mathcal{M}$ with $p \le p_0$, $\|(p_0 - p)v_i\| < \epsilon$ for all i, and and element $a \in \mathcal{A}$ such that $\|a\| \le \|bP_0\| + \epsilon$ and $bp = ap$.

Corolary 2.59. *Let $v \in \mathcal{H}$, $b \in \mathcal{M}$. Then there exist a sequence $(a_k) \subseteq \mathcal{A}$ such that $a_k v \to bv$ when $k \to \infty$.*

Proof. Take p_k a sequence of projections cooverging to identity and apply Luzin's theorem. □

2.4. **Representations of C^*-algebras.** This is the last subsection of preliminaries. It deals with the representations of a C^*-algebra. The main theorem here is Theorem 2.64 which gives a canonical way to buid representations of a C^*-algebra called the *Gelfand-Naimark-Segal* construction. This concept will be very helpful in defining definable closures and forking between types. References used here are [12] and [17].

Definition 2.60. Let $\mathcal{A}$ be a C^*-algebra. A *representation* is an algebra homomorphism $\pi : \mathcal{A} \to B(H)$ such that for all $a \in \mathcal{A}$, $\pi(a^*) = (\pi(a))^*$. In this case H is called an $\mathcal{A}$-*module*. A Hilbert subspace $H' \subseteq H$ is called an A-submodule of H if H' is close under π. H is called A-*irreducible* if H has no proper non trivial A-submodules. The set of repesentations of an algebra $\mathcal{A}$ is denoted $rep(\mathcal{A}, B(H))$.

Definition 2.61. Let G be a group of bounded operators on H. H is called a G-*module*. A Hilbert subspace $H' \subseteq H$ is called an G-submodule of H if H' is close under the action of G. H is called G-*irreducible* if H has no proper non trivial G-submodules.

Fact 2.62. Let G be the group of all the unitary operators on H. Then H is an irreducible G-module.

Proof. Given v,$w \in H$ such that $\|v\| = \|w\|$, there exists a unitary operator U such that $Uv = w$. □

Definition 2.63.

- Let (π, H) a representation of a C^*-algebra $\mathcal{A}$. (π, H) is called *non-degenerate* if for every nonzero vector $v \in H$, there exists $a \in \mathcal{A}$ such that $\pi(a)(v) \neq 0$. (π, H) is called *cyclic* if there exists a vector v_π such that $\pi(\mathcal{A})v_\pi$ is dense in H. Such a vector is called a *cyclic vector* for the representation (π, H).
- Let (p_i, H_i) for $i \in I$ a family of representations of $\mathcal{A}$. We define a representation $\oplus\pi_i$ on $\oplus H_i$ on this way: Let $v = \sum_i v_i$ and $a \in \mathcal{A}$, $\oplus\pi_i(a)v = \sum_i \pi_(a)v_i$.

Theorem 2.64 (Theorem 3.3.3. and Remarks 3.4.1. in [17])**.** *Let ϕ be a positive functional on $\mathcal{A}$. Then there exists a cyclic representation (π_ϕ, H_ϕ) with a cyclic vector $v_\phi \in H_\phi$ tal que for all $a \in \mathcal{A}$, $\phi(a) = \langle \pi_\phi(a)v_\phi | v_\phi \rangle$. This representation is called the Gelfand-Naimark-Segal construction. In the case when $\mathcal{A}$ is abelian and has a unit, $H_\phi = L^2(Sp(\mathcal{A}), \mu_{v_\phi})$, μ_{v_ϕ} is the radon measure corresponding to ϕ, v_ϕ is the identity function and π_ϕ is the morphism that takes $\mathcal{C}_0(Sp(\mathcal{A}), \mathbb{C})$ into the multiplication operators on $L^2(Sp(\mathcal{A}), \mu_{v_\phi})$.*

Definition 2.65. Two representations (π_1, H_1) and (π_2, H_2) are said *unitarily equivalent* if there exists an isometry U from H_1 to H_2 such that for every $a \in \mathcal{A}$, $U\pi_1(a)U^* = \pi_2(a)$.

Theorem 2.66 (Proposition 3.3.7 in [17])**.** *Two cyclic representations (π_1, H_1) and (π_2, H_2) are unitarily equivalent by mean of an isometry U such that $Uv_{\pi_1} = v_{\pi_2}$ if and only if for all $a \in \mathcal{A}$, $\langle \pi_1(a)v_{\pi_1} | v_{\pi_1} \rangle = \langle \pi_2(a)v_{\pi_2} | v_{\pi_2} \rangle$*

Definition 2.67. Two representations (π_1, H_1) and (π_2, H_2) are said *aproximately unitarily equivalent* if there exists a sequence of unitary operators U_n from H_1 to H_2 such that for every $a \in \mathcal{A}$, and every x, $y \in H_2$ $\langle U_n \pi_1(a) U_n^* x \mid y \rangle = \langle \lim \pi_2(a) \mid y \rangle$.

Theorem 2.68 (Theorem II.5.8 in [12])**.** *Two representations (π_1, H_1) and (π_2, H_2) are aproximately unitarily equivalent if and only if for all $a \in \mathcal{A}$, rank$(\pi_1(a)) =$ rank(π_2)*

Definition 2.69. We define the *discrete part* of $\mathcal{A}$, denoted by $\mathcal{A}_d$, the norm closure of the set of $a \in \mathcal{A}$ such that rank(a) is finite.

Fact 2.70. $\mathcal{A}_d$ is the ideal of compact operators in $\mathcal{A}$.

Definition 2.71.

- Let $\mathcal{A}$ be an abelian C^*-subalgebra of $B(H)$. The *discrete spectrum* of $\mathcal{A}$ $(Sp_d(\mathcal{A}))$ is the spectrum of the algebra $\mathcal{A}_d$.
- We define the *essential part* of $\mathcal{A}$, denoted by $\mathcal{A}_d$, the algebra $\mathcal{A}/\mathcal{A}_d$. The *essential spectrum* of $\mathcal{A}$ is the spectrum of the algebra $\mathcal{A}_e$.

Remark 2.72 (Generalized Weyl's criterion for essential spectrum). From previous definition it follows that if $t \in Sp(\mathcal{A})$, then $t \in Sp_e(\mathcal{A})$ if and only if, for every open neighborhood $V \subseteq Sp(\mathcal{A})$ of t, $\dim(\mathcal{H}_V) = \infty$

Lemma 2.73. *Suppose $\mathcal{A} = C_0(Sp(\mathcal{A}))$. Let $f \in \mathcal{A}$ and let $V \subseteq Sp(\mathcal{A})$ an open set contained supp(f). Then* $\dim(H_V) \leq$ *rank(f).*

Proof. Let $f \in \mathcal{A}$. $f \restriction H_V : H_V \to H$ has trivial kernel because $\text{supp}(f) \subseteq \text{supp}(\mu_v)$ for every $v \in H_V$. □

Theorem 2.74. *Two representations (π_1, H_1) and (π_2, H_2) of an abelian C^*-algebra $\mathcal{A}$ are aproximately unitarily equivalent if and only if the following holds:*

(1) $Sp(\pi_1(\mathcal{A})) = Sp(\pi_2(\mathcal{A}))$
(2) $Sp_e(\pi_1(\mathcal{A})) = Sp_e(\pi_2(\mathcal{A}))$
(3) *Given $t \in Sp(\mathcal{A})$, the dimension of the space of eigenvectors for t in H_1 is equal to the corresponding dimension in H_2*

Proof. Immediate consecuence from previous Lemma. □

Theorem 2.75 (Corollary 3.3.8. in [17])**.** *Let ϕ and ψ be positive functionals on $\mathcal{A}$. If $\phi \leq \alpha\psi$ for some $\alpha \in \mathbb{C}$ then (π_ϕ, H_ϕ) is unitarily equivalent to a subrepresentation of (π_ψ, H_ψ).*

Definition 2.76.

- A subset $F \subseteq S(\mathcal{A})$ is called *separating* if for every $a \in \mathcal{A}$ $\phi(a) = 0$ for every $\phi \in F$ implies that $a = 0$, i.e. the set $\{\phi\}$ is separating.
- Let $\phi \in S(\mathcal{A})$. ϕ is said to be *faithful* if for every $a \in \mathcal{A}_+$, $\phi(a) = 0$ implies that $a = 0$. A *faithful representation* is a representation (π, H) such that if $\pi(a) = 0$ then $a = 0$ for $a \in \mathcal{A}_+$.

Notation 2.77. For each $\phi \in S$, let $(\pi_\phi, H_\phi, v_\phi)$ be the Gelfand-Naimark-Segal construction of ϕ. For $F \subseteq S$ let $H_F = \oplus_{\phi\in F} H_\phi$ and $\pi_F = \oplus_{\phi\in F}\pi_\phi$.

Theorem 2.78 (Proposition 3.7.4 in [17])**.** *If $F \subseteq S$ is separating, then (π_F, H_F) is a faithful representation.*

Definition 2.79. The representation (π_S, H_S) is called the *universal representation*

3. Types and measures

As it was said in the introducction, denote by $\tilde{\mathcal{H}} = (\tilde{H}, \tilde{\mathcal{A}})$ an elementary extension of $(\mathcal{H}, \mathcal{A})$ which is saturated and homogeneous.

Definition 3.1. Let $\mathcal{A}$ a an abelian C^*-algebra of operators on a Hilbert space H. Let $V \subseteq Sp(\mathcal{A})$ a borel set. We denote by $\mathcal{H}_V$ the Hilbert subspace of H generated by all $v \in H$ such that $\text{supp}(\mu_v) \subseteq V$. In the same way, if $t \in Sp(\mathcal{A})$, then $\mathcal{H}_t$ will be the Hilbert subspace of H generated by the vectors $v \in \mathcal{H}$ such that $\mu_v = \delta_t$. Such vectors will be called *eigenvectors* for t.

Proposition 3.2. *An automorphism U of $(H, \mathcal{A})$ is a unitary operator U on H such that $US = SU$ for every $S \in \mathcal{A}$*

Proof. It is clear that U must be a linear operator. Also, we have that for every $v, w \in \mathcal{H}$ and $S \in \mathcal{A}$, we must have that $U(Sv) = S(Uv)$ and $\langle Uv \,|\, Uw\rangle = \langle v \,|\, w\rangle$ by definition of automorphism. Therefore U must be unitary. □

Remark 3.3. If S is a definable operator on $(H, \mathcal{A})$ and U is an automorphism of $(H, \mathcal{A})$, then $SU = US$ by definability.

Lemma 3.4. *Let S be a definable bounded normal operator in $\tilde{\mathcal{H}}$. Then for every $t \in Sp(\mathcal{A})$ $T \restriction \tilde{\mathcal{H}}_t = \alpha_t Id_{\tilde{\mathcal{H}}_t}$ for some $\alpha_t \in \mathbb{C}$.*

Proof. Let $t \in Sp(\mathcal{A})$ and U be an automorphism of $\tilde{\mathcal{H}}_t$. We have that $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_t \oplus \tilde{\mathcal{H}}_t^{\perp}$. Let $\tilde{U} = U \oplus Id_{\tilde{\mathcal{H}}_t^{\perp}} \in \mathrm{Aut}(\tilde{\mathcal{H}}_t)$. By Remark 3.3, T commutes with $\tilde{U}$, and therefore, $T \restriction \mathcal{H}_t : \tilde{\mathcal{H}}_t \to \tilde{\mathcal{H}}_t$ commutes with U. Then $T \restriction \tilde{\mathcal{H}}_t$ commutes with every automorphism of $\tilde{\mathcal{H}}_t$, and by Schur's lemma, there exists a complex number α_t such that $T \restriction \tilde{\mathcal{H}}_t = \alpha_t Id_{\tilde{H}_t}$ □

Theorem 3.5. *The bounded normal definible operators over $(H, \mathcal{A})$ are exactly the ones in $\mathcal{A}$.*

Proof. Let $\tilde{\mathcal{H}} = (\tilde{H}, \tilde{A})$ be as above. Let S be a definable bounded normal operator in $\tilde{\mathcal{H}}$. By previous Lemma, given $t \in Sp(\mathcal{A})$, there exists $\alpha_t \in \mathbb{C}$ such that $T \restriction \tilde{\mathcal{H}}_t = \alpha_t Id_{\tilde{\mathcal{H}}_t}$. Let $\omega : S(\mathcal{A}) \to \mathbb{C}$ be defined by $\omega(t) = \alpha_t$.

Suppose ω is not continuous an let t_0 a point of discontinuity. Let $(t_k)_{k\in\mathbb{N}}$ a sequence in $Sp(\mathcal{A})$ and $\mathcal{U}$ an ultrafilter over $\mathbb{N}$ such that $\lim_{\mathcal{U}} t_k = t_0$ but $\lim_U \omega(t_k) \neq \omega(t_0)$. There exist models $\mathcal{H}_k$ and $v_k \in \mathcal{H}_k$ such that $\mu_{v_k} = \delta_{t_k}$. Let $\mathcal{H} = \Pi_{\mathcal{U}} \mathcal{H}_k$ and let $v = (v_k)_{\mathcal{U}} \in \mathcal{H}$. It is easy to prove, via positive linear functionals, that $\mu_v = \delta_{t_0}$ and we have:

$$\omega(t_0) = \int_{Sp(\mathcal{A})} \omega d\mu_v = \Big(\int_{Sp(\mathcal{A})} \omega d\mu_{v_k} \Big)_{\mathcal{U}} = (\omega(t_k))_{\mathcal{U}} = \lim_{U} \omega(t_k)$$

which is a contradiction. □

The following is a straightforward conclusion from this theorem:

Corolary 3.6. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are abelian C^*-subalgebras of $B(H)$ such that $\mathcal{A}_1 \setminus \mathcal{A}_2 \neq \emptyset$, then there exists a unitary operator on H such that commutes with every element of $\mathcal{A}_2$ but it does not commute with every element in $\mathcal{A}$.*

Another corollary:

Corolary 3.7. *If $\tau(x)$ is an L-term, then there exist $S \in \mathcal{A}$ such that $\tau(x) = S(x)$.*

Proof. If $\mathcal{A}$ is abelian and $\tau(x)$ is an L-term, τ defines a bounded normal operator $S : H \to H$ such that $Sv = \tau(v)$ for $v \in H$. By Theorem 3.5 the conclusion follows. □

Corolary 3.8. *Let $v \in \mathcal{H}$ and let $S \in \mathcal{M}$ the strong closure of $\mathcal{A}$. Then $Sv \in dcl(v)$*

Proof. By previous Corollary 2.59. □

Definition 3.9. Given $v \in \mathcal{H}$, let $\mathcal{H}_v$ be the Hilbert subspace generated by Sv with $S \in \mathcal{M}$.

Theorem 3.10. *Let $v \in \mathcal{H}$. Then, v is a cyclic vector for $\mathcal{A}$ on $\mathcal{H}_v$.*

Proof. The set

$$\{\tau(v) \,|\, \tau(x) \text{ is an } L\text{-term in } s\}$$

is dense in $\mathcal{H}_v$ by Corollary 3.7 and Theorem 2.59 □

Corolary 3.11. *Let $v \in \mathcal{H}$. Then, $\mathcal{H}_v \simeq L^2(Sp(\mathcal{A}), \mu_v)$.*

Proof. By 2.64. □

Theorem 3.12. *Let v, $w \in \mathcal{H}$. Then $tp(v/\emptyset) = tp(w/\emptyset)$ if and only if $(\mathcal{H}_v, v)$ is isometrically isomorphic to $(\mathcal{H}_w, w)$.*

Proof. Let's suppose that for some $tp(v/\emptyset) = tp(w/\emptyset)$. Then

$$\{\|\tau(x)\| = \|\tau(v)\| \mid \tau(x) \text{ is an } L\text{-term in } x\} = \{\|\tau(x)\| = \|\tau(w)\| \mid \tau(x) \text{ is an } L\text{-term in } x\}$$

By Theorems 3.10 and 2.66 the function U such that $U(v) = w$ is an isomorphic isometry

Conversely, let $(\mathcal{H}_v, v)$ be isometrically isomorphic to $(\mathcal{H}_w, w)$. Let $\mathcal{H}_v^\perp$ and $\mathcal{H}_w^\perp$ be the closed orthogonal complements of $\mathcal{H}_v$ and $\mathcal{H}_w$ respectively in $\tilde{\mathcal{H}}$. Then we also have $\mathcal{H}_u^\perp \simeq \mathcal{H}_v^\perp$ and thus we get an automorphism of $\tilde{H}$ that sends v to w. □

Corolary 3.13. *Let $v, w \in \mathcal{H}$. Then $tp(v/\emptyset) = tp(w/\emptyset)$ if and only if $\mu_v = \mu_w$ if and only if $\phi_v = \phi_w$, where ϕ_v denotes the positive linear functional on $\mathcal{A}$ defined by v.*

Theorem 3.14. *Let $v, w \in \mathcal{H}$. Then $\mathcal{H}_v$ is isometrically isomorphic to a Hilbert subspace of $\mathcal{H}_w$ if and only if $\mu_v \ll \mu_w$.*

Proof. By Theorems 2.75 and 2.55, if $\mu_u \ll \mu_v$ then $\mathcal{H}_v$ is isometrically equivalent to a Hilbert subspace of $\mathcal{H}_w$. The converse comes from Theorem 2.75. □

Definition 3.15. Given $A \subseteq \mathcal{H}$, let $\mathcal{H}_A$ be the Hilbert subspace of $\mathcal{H}$ generated by the elements Sa, where $a \in A$ and $S \in \mathcal{M}$. With the same asumptions, let $\mathcal{H}_A^\perp$ the orthogonal complement of $\mathcal{H}_A$ and P_A and $P_{A^\perp}$ denote the projections over $\mathcal{H}_A$ and $\mathcal{H}_A^\perp$ respectively.

Lemma 3.16. *Let $v \in \mathcal{H}$ and $A \subseteq \mathcal{H}$. If $v \perp \mathcal{H}_A$ then $\mathcal{H}_v \perp \mathcal{H}_A$.*

Proof. Let $S_1, S_2 \in \mathcal{M}$ and let $a \in A$. Then

$$\langle S_1 v \mid S_2 a\rangle = \langle v \mid S_1^* S_2 a\rangle = \langle S_2^* S_1 v \mid a\rangle = 0$$

. □

Theorem 3.17. *Let $v \in \mathcal{H}$ and $A \subseteq \mathcal{H}$. Then*

$$tp(v/A) = tp(P_{A^\perp}(v)/\emptyset) \cup [(x - P_A(v)) \perp \mathcal{H}_A,]$$

where $(x - P_A(v)) \perp \mathcal{H}_A$ is the set of conditions $\{|\langle x \mid a\rangle - \langle P_A(u) \mid a\rangle| = 0 \mid a \in \mathcal{H}_A\}$

Proof. Suppose $u, v \in \mathcal{H}$ are such that

$$tp(P_{A^\perp}(u)/\emptyset) \cup (x - P_A()) \perp \mathcal{H}_A = tp(P_{A^\perp}(v)/\emptyset) \cup (x - P_A(v)) \perp \mathcal{H}_A$$

Then $P_A(u) = P_A(v)$ and there exists an automorphism of the moster model that takes $P_{A^\perp}(u)$ to $P_{A^\perp}(v)$ and. Let g be that automorphism. By previous Lemma we have that $g(\mathcal{H}_{P_{A^\perp}(u)}) = \mathcal{H}_{P_{A^\perp}(v)}$. So, taking suitable basis for $\tilde{H}$, we can build an automorphism g' of the moster model that $g' \restriction \mathcal{H}_A = \text{id}$ and $g' \restriction \mathcal{H}_A^\perp = g \restriction \mathcal{H}_A^\perp$ □

Corolary 3.18. *The structure $\mathcal{H} = (H, \mathcal{A})$ admits quantifier elimination.*

Proof. Theorems 3.12 and 3.17 shows that the quantifier free type of an element determines the the type of that element. □

Corolary 3.19. *Let $A \subseteq \mathcal{H}$. Then $dcl(A) = \mathcal{H}_A$*

Proof. From Corollary 3.8, it is clear that $\mathcal{H}_A \subseteq dcl(A)$. For the converse, let $v \notin \mathcal{H}_A$. Then $P_{A^\perp}(v) \neq 0$. Let $u \in \mathcal{H}$ such that $u \neq P_{A^\perp}(v)$, $u \perp \mathcal{H}_A$ and $\|u\| = \|P_{A^\perp}(v)\|$. Then by Lemma 3.16, $\mathcal{H}_u \perp \mathcal{H}_A$ and by Theorem 3.17 $tp(P_A(v) + u/A) = tp(v/A)$. But there are several such $u \in \mathcal{H}$, so $v \notin dcl(A)$. □

Corolary 3.20. *Let $A \subseteq \mathcal{H}$. Then $U \in Aut(\mathcal{H}/A)$ if and only if U keeps $\mathcal{H}_A$ fixed.*

Corolary 3.21. *Let $A \subseteq B$, $p \in S(A)$ $q \in S(B)$ and $v, w \in \mathcal{H}$ such that $p = tp(v/A)$ and $q = tp(w/B)$. Then q is an extension of p if and only if the following condition hold:*

(1) $P_{acl(A)}v = P_{acl(A)}w$

(2) $(\mathcal{H}_{P^{\perp}_{acl(A)}v}, P^{\perp}_{acl(A)}v)$ *is isometrically isomorphic to* $(\mathcal{H}_{P^{\perp}_{acl(A)}w}, P^{\perp}_{acl(A)}w)$.

Definition 3.22. We say that a set $A \subseteq H$ is *sequentially independent* if no element of A is the limit of a sequence of linear combinations of elements in A different from itself.

Theorem 3.23. *Let A be a sequentially independent set. Then*

$$\mathcal{H}_A \simeq \oplus_{v\in A}\mathcal{H}_v \simeq \oplus_{v\in A}L^2(Sp(\mathcal{A}), \mu_v)$$

Proof. By sequential independence and Corollary 3.11. □

Corolary 3.24. *Let $v \in \mathcal{H}$. Then*

$$dcl(v) = \{Tv \,|\, T \in \mathcal{M}\}$$

Proof. By Corollary 3.19 □

4. The theory of $\mathcal{H}$

In this section we use some results from section 2 to axiomatize $Th(\mathcal{H})$ in the context of continous logic. The main tool here is the Theorem 2.68 which is mainly a consecuence of Voiculescu's theorem (see [12]). Finally, we show that this theory admits quantifier elimination.

Theorem 4.1. *Let $S \in \mathcal{A}$. Then $rank(S) \geq n \in \mathbb{N}$ if and only if for every $r \in \mathbb{Q}+$*

$$\mathcal{H} \models \inf_{u_1 u_2 \cdots u_n} \max\big(|\langle S_H u_i | S_H u_j\rangle|, |\|S_H u_i\| - 1|\big) = 0 \tag{3}$$

Proof. This condition is a continuous logic condition for:

$$\exists u_1 u_2 \cdots u_n \forall i(|\|Su_i\| = 1) \wedge \forall i \neq j(\langle Su_i | Su_j\rangle = 0)$$

i.e this condition says that there are n orthonormal vectors in the image of S □

Theorem 4.2. *Let $S \in \Sigma$. Then $rank(S) = n \in \mathbb{N}$ if and only if for every $\epsilon > 0$*

(4)

$$\inf_{u_1 u_2 \cdots u_n} \sup_v \max\big(|\langle S_H u_i | S_H u_j\rangle|, |\|S_H u_i\| - 1|, \|Sv - \sum_{k=1}^{m} \langle Sv | Su_i\rangle u_i)\| \dot{-} \epsilon\big) = 0$$

Proof. This condition is a continuous logic condition for:

$$\exists u_1 u_2 \cdots u_n \forall i(|\|Su_i\| = 1) \wedge \forall i \neq j(\langle Su_i | Su_j\rangle = 0) \wedge \forall v(Sv = \sum_{k=1}^{m} \langle Sv | Su_i\rangle u_i) = 0$$

i.e. it says that $\text{Im}(S)$ has a basis of size n. □

Next theorem is a remark from C.W. Henson:

Theorem 4.3. *Let $\mathcal{A}$ an abelian C^*-algebras of operators on the separable Hilbert space $\mathcal{H}$, and π_1 and π_2 two representations of $\mathcal{A}$ on H. Then the structures $(H, \pi_1(A))$ and $(H, \pi_2(B))$ are elementarily equivalent if and only if π_1 and π_2 are aproximately unitarily equivalent.*

Proof. $\Rightarrow$ Suppose $(\mathcal{H}, \pi_1(\mathcal{A})) \equiv (H, \pi_2(\mathcal{A}))$. Then by if $S \in \mathcal{A}_d$ and $n = \mathrm{rank}(S)$ then Condition 4 will hold for every $\epsilon > 0$. In the same manner, if $S \in \mathcal{A}_e$, Equation 3 will hold for every n and the hypotesis of Theorem 2.68 will hold, and therefore π_1 and π_2 are aproximately unitarily equivalent.

$\Leftarrow$ Suppose $\pi_1(\mathcal{A})$ and $\pi_2(B)$ are aproximately unitarily equivalent. Then, there exists a sequence of unitary operators U_n such that for every $x, y \in H$ and $S \in \mathcal{A}$, $\lim_{n\to\infty}\langle U_n\pi_1(S)U_n^* x|y\rangle = \langle \pi_2(ASx|y\rangle$. Let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$ which contains the filter of cofinite subsets of $\mathbb{N}$. Let $\mathcal{H}_1 = (H_1, \mathcal{A}_1) = \Pi_{\mathcal{U}}(\mathcal{H}, U_n\pi_1(A)U_n^*)$ and let $\mathcal{H}_2 = (H_2, \mathcal{A}_2) = \Pi_{\mathcal{U}}(H, \pi_2(\mathcal{A}))$. It follows that $\mathcal{H}_1 \simeq \mathcal{H}_2$ and by Keisler-Shelah's theorem, $(\mathcal{H}, \pi_1(A)) \equiv (H, \pi_2(B))$.

□

Definition 4.4. Let $T_{\mathcal{A}}$ the theory of Hilbert spaces together with the following conditions:

(1) For $v \in H_1$ and $S, T \in \mathcal{A}_1$:
$$(ST)v = S(Tv)$$
(2) For $v \in H_1$ and $S, T \in \mathcal{A}_1$:
$$\tfrac{S+T}{2}(v) = \tfrac{Sv+Tv}{2}$$
(3) For $v \in H_1$ and $S \in \mathcal{A}_1$:
$$\langle Sv \mid w\rangle = \langle v \mid S^* w\rangle$$
(4) For $v \in H_1$ and $S \in \mathcal{A}_1$:
$$\|Sv\| \leq \ |S\|\|v\|$$
(5) For $v \in H_1$ and Id the identity operator in $\mathcal{A}$:
$$(iId)v = iv$$
(6) For $v, w \in H_1$, and $S \in \mathcal{A}_1$:
$$S\big(\tfrac{v+w}{2}\big) = \tfrac{Sv+Sw}{2}$$
(7) For $S \in \mathcal{A}_e \cap \mathcal{A}_1$, and every $n \in \mathbb{N}$:
$$\exists u_1 u_2 \cdots u_n \forall i(|\|Su_i\| = 1) \wedge \forall i \neq j(\langle Su_i|Su_j\rangle = 0)$$
(8) Let $S \in \mathcal{A}_d \cap \mathcal{A}_1$, $\mathrm{rank}(S) = n \in \mathbb{N}$,
$$\exists u_1 u_2 \cdots u_n \forall i(|\|Su_i\| = 1) \wedge \forall i \neq j(\langle Su_i|Su_j\rangle = 0) \wedge \forall v(Sv = \sum_{k=1}^{m}\langle Sv|Su_i\rangle u_i) = 0$$

Remark 4.5. We give the complete continuous logic formalism only for the last two conditions which are the most interesting ones (see Theorems 4.1 and 4.2). The traslations for the other conditions to the continuous logic formalism is straigforward and is left to the reader.

Corolary 4.6. *$T_{\mathcal{A}}$ axiomatizes the theory $Th(\mathcal{H})$.*

Proof. By Theorem 4.3. □

Lemma 4.7. *Let $v \in \mathcal{H}$. If v is a eigenvector corresponding to some $t \in Sp_d(\mathcal{A})$ then v is algebraic over $\emptyset$.*

Proof. By definition $t \in Sp_d(\mathcal{A})$ if and only if t is isolated in $Sp(\mathcal{A})$ with finite dimensional eigenspace $\mathcal{H}_t$. So any automorphism can only send $\mathcal{H}_t$ onto $\mathcal{H}_t$ and the orbit of v under such automorphism can only be compact. □

5. Algebraic vectors

Lemma 5.1. *Let $v \in \mathcal{H}$ be such that $v = \sum v_k$ where each v_k is an eigenvector for some $t_k \in Sp_d(\mathcal{A})$. Then v is algebraic over $\emptyset$.*

Proof. Given that $\|v_k\| \to 0$ when $k \to \infty$, the orbit of v under all the automorphisms is a Hilbert cube which is compact. □

Lemma 5.2. *Let $v \in \mathcal{H}$. Then v is algebraic over $\emptyset$ if and only if for every $t \in Sp_e(\mathcal{A})$ there exists an open neighborhood of t, $V \subseteq Sp(\mathcal{A})$, then $v \perp \mathcal{H}_V$.*

Proof.
- **:** $\Rightarrow$ Suppose $v \not\perp \mathcal{H}_V$ for every neighborhood V of $t \in Sp_e(\mathcal{A})$. By Remark 2.72 for every open subset V of $Sp_e(\mathcal{A})$ we have that $\dim(\mathcal{H}_V) = \infty$. For every V open subset of $Sp_e(N)$, let v_V^k be a sequence of mutually orthogonal elements in $\mathcal{H}_V$ such that for all k, $\|v_V^k\| = \|P_{\mathcal{H}_V} v\|$. Let $I = \{i \mid i$ is a finite collection of disjoint open subsets of $Sp(\mathcal{A})\}$. For $i \in I$, let $w_i^k = \sum_{V \in i} v^k_{V \cap Sp_e(\mathcal{A})} + \chi_{V \cup Sp_d(\mathcal{A})} v$. Let $\mathcal{U}$ be an ultrafilter in I. Let $\mathcal{H}' = \Pi_{I,\mathcal{U}} \mathcal{H}$ Let $\bar{w}^k = (w_i^k)_{I,\mathcal{U}} \in \mathcal{H}'$. Then for all k, bw^k has the same type of v over $\emptyset$ so the orbit of v is infinite dimensional and v is not algebraic over $\emptyset$.
- **:** $\Leftarrow$ Saying that $v \perp \mathcal{H}_V$ for some open neighborhood V of $tSp(\mathcal{A})$ is equivalent to saying that $v = \sum v_n$ with v_n eigenvectors for $t_n \in Sp_d(\mathcal{A})$ and by Lemma 5.1 v is algebraic over the $\emptyset$.

□

Corolary 5.3. *Let $v \in \mathcal{H}$. Then v is algebraic over $\emptyset$ if and only if $supp(\mu_v) \cap Sp_e(\mathcal{A}) = \emptyset$.*

Theorem 5.4. *Let $A \subseteq \mathcal{H}$. Then $acl(A)$ is closed Hilbert subspace generated by the union of $dcl(A)$ with $acl(\emptyset)$.*

Proof. Let E be the space $acl(\emptyset) + dcl(A)$. We have that $acl(\emptyset) \subseteq acl(A)$ and $dcl(A) \subseteq acl(A)$ so $E \subseteq acl(A)$. If $v \notin E$, let $P_E v$ be the projection of v onto E. We have that $w - P_E w \neq 0$ sofor some borel subset V of $Sp(\mathcal{A})$, $\chi_V v - \chi_v P_E \neq 0$. We can assume V is an open subset of $Sp_e(\mathcal{A})$, so by Remark 2.72 we can find an automorphism that takes v into a non compact orbit in $\mathcal{H}$, so $v \notin acl(A)$. □

Lemma 5.5. *Let (X, μ) be a measure space and let $X = Y \cup Z$ where Y, Z are measurable subsets of X. If $Y \cap Z = \emptyset$ then for every $p \geq 1$ $L^p(X, \mu) \simeq L^p(Y, \mu) \oplus L^p(Z, \mu)$.*

Proof. Clear from the definition of L^p spaces. □

Definition 5.6. Let $\mathcal{H}_d = \oplus_{t \in Sp_d(\mathcal{A})} \mathcal{H}_t$. We call this space the *algebraic part* of $\mathcal{H}$.

Proposition 5.7. *$\mathcal{H}_d \simeq \bigoplus_{\sigma_d(N)} \mathbb{C}^{n_t}$, where $n_t \dim(\mathcal{H}_t)$.*

Proof. Clear. □

Theorem 5.8. *Let $v \in \tilde{\mathcal{H}}$. Then $\mathcal{H}_v \simeq \left(\bigoplus_{t\in Sp_d(\mathcal{A})} \mathbb{C}\mu_v(t)\right) \oplus L^2(Sp_e(\mathcal{A}), \mu_v)$*

Proof. By Theorem 3.11, $\mathcal{H}_v \simeq L^2(Sp(\mathcal{A}), \mu_v)$. By Lemma 5.5, if $v \in \mathcal{H}$, then, $\mathcal{H}_v \simeq L^2(Sp_d(\mathcal{A}), \mu_v) \oplus L^2(Sp_e(\mathcal{A}), \mu_v) \simeq \left(\bigoplus_{t\in Sp_d(\mathcal{A})} L^2(\{t\}, \mu_v)\right) \oplus L^2(Sp_e(\mathcal{A}), \mu_v)$. On the other hand, $L^2(\{t\}, \mu_v) \simeq \mathbb{C}\mu(t)$ for every $t \in Sp_d(\mathcal{A})$. Then follows the conclusion. □

Definition 5.9. Previous theorem shows that for $v \in \tilde{\mathcal{H}}$, $\mathcal{H}_v = \mathcal{H}_v^d \oplus \mathcal{H}_v^e$ such that $\mathcal{H}_v^d \simeq \bigoplus_{t\in Sp_d(\mathcal{A})} \mathbb{C}\mu(t)$ and $\mathcal{H}_v^e \simeq L^2(Sp_e(\mathcal{A}), \mu_v)$. $\mathcal{H}_v^d$ is called the *algebraic part* of $\mathcal{H}_v$ while $\mathcal{H}_v^e$ is called the *essential part* of $\mathcal{H}_v$

Theorem 5.10. *Let $\kappa \geq |Sp(\mathcal{A})|$ such that $cf\kappa \neq \omega$. The structure*

$$\tilde{\mathcal{H}}_\kappa \simeq \Big(\bigoplus_{Sp_d(\mathcal{A})} \mathbb{C}^{n_t}\Big) \oplus \bigoplus_{\kappa} [\oplus_{\phi\in S(\mathcal{C}_0(Sp_e(\mathcal{A}))} L^2(Sp_e(\mathcal{A}), \mu_\phi)]$$

is κ saturated and κ homogeneous elementary superstructure of $\mathcal{H}$, and therefore, works as a monster model for $Th(\mathcal{H})$.

Proof. κ-saturation is given by Theorems 3.12, 3.17 and 3.23. κ- homogeneity is as follows: Let S be a partial isometric isomorphism between A, and $B \subseteq \tilde{\mathcal{H}}$. Let v, $w \in \tilde{\mathcal{H}}$ such that $tp(v/A) = tp(w/B)$. Then, by Theorems 3.12 and 3.17 $\mathcal{H}_{P_A^\perp(v)}$ is isometrically isomorphic to $\mathcal{H}_{P_B^\perp(w)}$. By Theorems 3.11 and 3.23, the operator S' that extends S over $\mathcal{H}_A$ and such $T'(v) = w$ is isometric. □

6. Stability and forking

Theorem 6.1. *Let $p, q \in S(\emptyset)$ and let $v, w \in \tilde{\mathcal{H}}$ such that $v \models p$ and $w \models q$, and $\mu_v \ll \mu_w$. Then, $d(p, q) = \|\mu_w - \mu_v\|$*

Proof. If $\mu_u \ll \mu_v$, by Theorem 3.14, there exist $v' \models tp(v/\emptyset)$ such that $\mathcal{H}_{v'} \leq \mathcal{H}_w$ and there exists $f \in L^1(Sp(\mathcal{A}), \mu_w)$ such that $d\mu_v = fd\mu_w$. Then $d|\mu_w - \mu_v| = |1 - f|d\mu_w$ and therefore $d(p, q) = \|\mu_w - \mu_v\|$. □

Theorem 6.2. *Let $p, q \in S(\emptyset)$ and let $v, w \in \tilde{\mathcal{H}}$ be such that $v \models p$ and $w \models q$, and $\mu_v \perp \mu_w$. Then, $d(p, q) = \sqrt{\|\mu_v\|^2 + \|\mu_w\|^2}$*

Proof. If $\mu_v \perp \mu_w$, by Theorem 3.14, neither $\mathcal{H}_v$ is not isometrically isomorphic to a Hilbert subspace of $\mathcal{H}_w$ nor $\mathcal{H}_w$ is isometrically isomorphic to a Hilbert subspace of $\mathcal{H}_v$. Then we can assume $\mathcal{H}_v \perp \mathcal{H}_w$ and therefore, $d(p, q) = \|v - w\| = \sqrt{\|v\|^2 + \|w\|^2} = \sqrt{\|\mu_v\|^2 + \|\mu_w\|^2}$. □

Theorem 6.3. *Let $p, q \in S(\emptyset)$ and let $v, w \in \tilde{\mathcal{H}}$ be such that $v \models p$ and $w \models q$, and $\mu_w = \mu_w^\parallel + \mu_w^\perp$ according to Lebesgue decomposition theorem. Then, $d(p, q) = \sqrt{\|\mu_v - \mu_w^\parallel\|^2 + \|\mu_w^\perp\|^2}$*

Proof. By Theorems 6.1 and 6.2. □

Theorem 6.4. *Let $p, q \in S(A)$ and let $v, w \in \mathcal{H}$ be such that $u \models p$ and $v \models q$. Then,*

$$d(p, q) = \sqrt{[P_A(v) - P_A(w)]^2 + d^2(tp(P_A^\perp(v)/\emptyset), tp(P_A^\perp(w)/\emptyset)}$$

Proof. By Theorems 3.17 □

Remark 6.5. If X is any topological space, we denote by den(X) its *density*, i.e. the least cardinal of a dense set in X. If A is a set but not a topological space, we denote by $|A|$ its cardinality.

Corolary 6.6. *Let $A \subseteq \mathcal{H}$ then $den(S_1(A)) \leq |A| \times 2^{\aleph_0}$*

Proof. Clear from Theorems 3.12, 3.17 and Corollary 6.4. □

Theorem 6.7. *The structure $\mathcal{H}$ is κ-stable for $\kappa \geq den(Sp(\mathcal{A}))$.*

Proof. Clear from Corollary 6.6. □

Definition 6.8. Let $v \in \mathcal{H}$ and $A \subseteq \mathcal{H}$. Then $P_{dcl(A)}(v)$ and $P_{acl(A)}(v)$ denote the projection of v onto the spaces $dcl(A)$ and $acl(A)$ repectively. In the same manner, $P_{dcl(A)^\perp}(v)$ and $P_{acl(A)^\perp}(v)$ denote the projection of v onto the spaces $dcl(A)^\perp$ and $acl(A)^\perp$ repectively.

Definition 6.9. Let v, $w \in \mathcal{H}$. We say that v is *independent* from w over $\emptyset$ if $\mathcal{H}_v \perp \mathcal{H}_w$ and denote it $v \mathop{\smile\!\!\!\!\!\!|}^*_\emptyset w$.

Definition 6.10. Let v, $w \in \mathcal{H}$. Let A. We say that v is *independent* from w over A if $\mathcal{H}_{P^\perp_{acl(A)}(v)} \perp \mathcal{H}_{P^\perp_{acl(A)}(w)}$ and denote it $v \mathop{\smile\!\!\!\!\!\!|}^*_A w$.

Fact 6.11. Independence over the empty set in the sense of Definition 6.9 and 6.10 coincide.

Definition 6.12. Let $v \in \mathcal{H}$. Let A, $B \subseteq \mathcal{H}$. We say that v is *independent* from B over A if $P_{acl(A)}(v) = P_{acl(A\cup B)}(v)$ and denote it $v \mathop{\smile\!\!\!\!\!\!|}^*_A B$.

Theorem 6.13. *Let $v, w \in \mathcal{H}$ and $A \subseteq \mathcal{H}$. Then $v \mathop{\smile\!\!\!\!\!\!|}^*_A w$ in the sense of 6.10 if and only if $\{v\} \mathop{\smile\!\!\!\!\!\!|}^*_A \{w\}$ in the sense of 6.12*

Proof. **⇐):** Suppose $\{v\} \mathop{\smile\!\!\!\!\!\!|}^*_A \{w\}$ in the sense of 6.12. Then $P_{acl(A)}(v) = P_{acl(A\cup\{w\})}(v)$. So $P_{acl(A)}(v) = P_{\overline{<acl(A)\cup\mathcal{H}_w>}}(v)$. Then $v - P_{acl(A)}(v) = v - P_{\overline{<acl(A)\cup\mathcal{H}_w>}}(v)$. But $v - P_{\overline{<acl(A)\cup\mathcal{H}_w>}}(v) \perp \mathcal{H}_w$. Then $v - P_{acl(A)}(v) \perp \mathcal{H}_w$ which implies that $v - P_{acl(A)}v \perp f(N)w$ for every $f \in \mathcal{B}(Sp(\mathcal{A}), \mathbb{C})$. This means that $P^\perp_{acl(A)}(v) \perp \mathcal{H}_w$. By 3.16 this means that $\mathcal{H}_{P^\perp_{acl(A)}(v)} \perp \mathcal{H}_w$ and therefore, $\mathcal{H}_{P^\perp_{acl(A)}(v)} \perp \mathcal{H}_{P^\perp_{acl(A)}(w)}$

⇒): Previous implications are reversible.

□

Theorem 6.14. *Let $A \subseteq B$, $p \in S(A)$ $q \in S(B)$ and $v, w \in \mathcal{H}$ such that $p = tp(v/A)$ and $q = tp(w/B)$. Then q is a non-forking extension of p if and only if the following condition hold:*

(1) $P_{acl(A)}v = P_{acl(B)}w$
(2) $(\mathcal{H}_{P^\perp_{acl(A)}v}, P^\perp_{acl(A)}v)$ *is isometrically isomorphic to* $(\mathcal{H}_{P^\perp_{acl(B)}w}, P^\perp_{acl(B)}w)$

Proof. Clear from Theorem 3.12 and Definition 6.10. □

Theorem 6.15. $\mathop{\smile\!\!\!\!\!\!|}^*$ *is a freeness relation.*

Proof. **Finite character:** We show that for $v \in \mathcal{H}$, $A, B \subseteq \mathcal{H}$, $v \mathop{\smile\!\!\!\!\!\!|}^*_A B$ if and only if $v \mathop{\smile\!\!\!\!\!\!|}^*_A B_0$ for every finite $B_0 \subseteq B$. The $\Rightarrow$) direction is clear. For the $\Leftarrow$) direction, suppose that $v \not\mathop{\smile\!\!\!\!\!\!|}^*_A B$. Let $w = P_{acl(A\cup B)}(v) - P_{acl(A)}(v)$. Then $w \in acl(A\cup B) \setminus acl(A)$. So there exist w_1, w_2 such that $w_1 \in dcl(A \cup B)$, $w_2 \in acl(\emptyset)$ and $w = w_1 + w_2$. By theorem 3.19 there exist sequences $(v_k)_{k\in\mathbb{N}} \subseteq A \cup B$ and f_k $k \in \mathbb{N}$ such that $\sum_{k=0}^{n} f_k(v_k) \to w_1$ when $n \to \infty$. Let E be the span of $(v_k)_{k\in\mathbb{N}}$. If $\dim(E) = \infty$ we are done. If not, let $N \in \mathbb{N}$ be such that $\|w_1 - P_{dcl(A\cup\{v_1,\cdots,v_N\})}(w_1)\|(\frac{\|w_1\|}{2}$ and at least for one $v_k \notin A$. Let $B_0 = B \cap \{v_1, \cdots, v_N\}$. Then $P_{A\cup B_0}(v) = P_{dcl(A\cup\{v_1,\cdots,v_N\})}(w_1) + w_2$ and $v \not\mathop{\smile\!\!\!\!\!\!|}^*_A B_0$

Local character: Let $v \in \mathcal{H}$ and $B \subseteq \mathcal{H}$. Let $w = P_{acl(B)}(v)$. Then there exists a sequence $(b_k)_{k\in\mathbb{N}} \subseteq B$ such that $b_k \to w$ when $k \to \infty$. Let $B_0 = \{b_k \,|\, k \in \mathbb{N}\}$. Then $v \mathop{\smile\!\!\!\!\!\!|}^*_{B_0} B$ and $|B_0| = \aleph_0$

Transitivity of independence: Let $v \in \mathcal{H}$ and $A \subseteq B \subseteq C \subseteq H$. If $v \mathop{\smile\!\!\!\!\!\!|}^*_A C$ then $P_{acl(A)}(v) = P_{acl(C)}(v)$. It is clear that $P_{acl(A)}(v) = P_{acl(B)}(v) = P_{acl(C)}(v)$ so $v \mathop{\smile\!\!\!\!\!\!|}^*_A B$ and $v \mathop{\smile\!\!\!\!\!\!|}^*_B C$. Conversely, if $v \mathop{\smile\!\!\!\!\!\!|}^*_A B$ and $v \mathop{\smile\!\!\!\!\!\!|}^*_B C$, we have that $P_{acl(A)}(v) = P_{acl(B)}(v)$ and $P_{acl(B)}(v) = P_{acl(C)}(v)$. Then $P_{acl(A)}(v) = P_{acl(C)}(v)$ and $v \mathop{\smile\!\!\!\!\!\!|}^*_A C$.

Symmetry: It is a clear from Definition 6.10.

Invariance: Let U be an automorphism of $(H, \mathcal{A})$. Let $v, w \in \mathcal{H}$ and $A \subseteq \mathcal{H}$ be such that $v \mathop{\smile\!\!\!\!\!\!|}^*_C w$. This means that $\mathcal{H}_{P^{\perp}_{acl(A)}(v)} \perp \mathcal{H}_{P^{\perp}_{acl(A)}(w)}$. It follows that $\mathcal{H}_{P^{\perp}_{acl(UA)}(Uv)} \perp \mathcal{H}_{P^{\perp}_{acl(UA)}(Uw)}$ and $Uv \mathop{\smile\!\!\!\!\!\!|}^*_{UC} Uw$.

Existence: Let $A \subseteq B \subseteq \tilde{H}$, $p \in S(A)$ and $v \in \mathcal{H}$ be such that $p = tp(v/A)$. Let $\mathcal{H}$ be a cyclic representation unitarily equivalent to $\mathcal{H}_{P^{\perp}_{acl(A)}(v)}$ and let $w \in \mathcal{H}$ be the cyclic vector corresponding to $P^{\perp}_{acl(A)}(v)$. Let $v' = P_{acl(A)}(v) + w$. Then $tp(w/B)$ is a non-forking extension of $tp(v/A)$.

Stationarity: It is clear by Theorem 6.14.

□

Definition 6.16. Let $v \in \mathcal{H}$ and $A \subseteq \mathcal{H}$. Let $Cb(v/A) = \{a_k \,|\, k \in \mathbb{N}\}$ be a sequence in A such that there exist a sequence $\{f_k \,|\, k \in \mathbb{N}\}$ in $\mathcal{B}(Sp(\mathcal{A}), \mathbb{C})$ such that $f_k(a_k) \to P_{dcl(A)}v$.

Theorem 6.17. *Let $v \in \mathcal{H}$ and $A \subseteq \mathcal{H}$. Then $Cb(v/A)$ is a canonical base for the type $tp(v/A)$*

Proof. By Theorem 6.14 $tp(v/A)$ does not fork over $Cb(v/A)$ which is countable. □

Theorem 6.18. *Let $p, q \in S_1(\emptyset)$, let $v \models p$ and $w \models q$. Then, $p \perp^a q$ if and only if $\mu_v \perp \mu_w$.*

Proof. $p \perp^a q$ if and only if $\mathcal{H}_{v'} \perp \mathcal{H}_{w'}$ for all $v' \models p$ and $w' \models q$. By Lesbesgue decomposition theorem $\mu_w = \mu_v^{\|} + \mu_v^{\perp}$ where, $\mu_v^{\|} \ll \mu_v$ and $\mu_v^{\perp} \perp \mu_v$. $\mu_v^{\|} \neq 0$ if and only if there is a choice of $v' \models p$ and $w' \models q$ such that $\mathcal{H}_{v'} \cap \mathcal{H}_{w'} \neq \{0\}$ and therefore $\mathcal{H}_{v'} \not\perp \mathcal{H}_{w'}$. □

Corolary 6.19. *Let $A \subseteq H$ be such that $A = acl(A)$. Let $p, q \in S_1(A)$, let $v \models p$ and $w \models q$. Then, $p \perp^a_A q$ if and only if $\mu_{P^{\perp}_{acl(A)}(v)} \perp \mu_{P^{\perp}_{acl(A)}(w)}$*

Proof. Clear from previous theorem. □

Corolary 6.20. *Let $A \subseteq H$ be such that $A = acl(A)$. Let $p, q \in S_1(A)$. Then, $p \perp^a q$ if and only if $p \perp q$.*

Theorem 6.21. *Let $p, q \in S_1(\emptyset)$, let $v \models p$ and $w \models q$. Then, $p \rhd_\emptyset q$ if and only if $\mu_w \ll \mu_v$.*

Proof. Suppose $p \rhd_\emptyset q$. Suppose that v and w are such that if $v \downarrow^*_\emptyset A$ then $w \downarrow^*_\emptyset A$ for every A. Then for every A if $\mathcal{H}_v \perp \mathcal{H}_A$ then $\mathcal{H}_w \perp \mathcal{H}_A$. This means $\mathcal{H}_w \subseteq \mathcal{H}_v$ and $\mathcal{H}_w$ is unitarily equivalent to some Hilbert subspace of $\mathcal{H}_w$ and by Theorem 3.14 $\mu_w \ll \mu_v$. □

Corolary 6.22. *Let A, $\mathfrak{B}$ be small subsets of $\tilde{H}$ and $p \in S_1(A)$ and $q \in S_1(B)$ two stationary types. Then $p \rhd_C q$ if and only if there exist v $w \in \tilde{\mathcal{H}}$ such that $tp(v/C)$ is a non-forking extension of p, $tp(w/C)$ is a non-forking extension of q and $\mu_{P^\perp_{acl(A)} w} \ll \mu_{P^\perp_{acl(A)} v}$.*

Proof. Clear from previous theorem. □

Let U be a vector space automorphism of monster model $\tilde{\mathcal{H}}$ and $\epsilon > 0$. Then U is called an *ϵ-automorphism* if for every $v \in \tilde{H}$ Two Hilber spaces $\mathcal{H}_1$, $\mathcal{H}_2 \prec \tilde{\mathcal{H}}$ are called *ϵ-isomorphic* if there exists an ϵ-automorphism U of $\tilde{\mathcal{H}}$ such that $U(\mathcal{H}_1) = U(\mathcal{H}_2)$.

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