

OPERATORS ALGEBRAS IN HILBERT SPACES

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ABSTRACT. We give a brief summary of C^* and Von Neumann algebra theory according to [10] with some additions from [3], [4], [11] and [12].

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1. BACKGROUND ABOUT C^* -ALGEBRAS

In this section we study the basic theory of C^* -algebras and their representations along with some important theorems of measure theory. We define positive linear functionals, states, representation and cyclic representations which are key notion for the work in next sections. The main sources of this section are [11] for elementary operator theory, [12] for measure theory and elementary analysis, [3] for spectral theory of operators and [4] and [10] for C^* -algebra theory.

1.1. C^* -algebras. In this subsection we give some elementary facts about C^* -algebras. References used here are [11], [4] and [10].

Definition 1.1. Let \mathcal{A} be a complex Banach algebra. \mathcal{A} is called a C^* -algebra if there exists a map $*$: $\mathcal{A} \rightarrow \mathcal{A}$, called *involution* such that for all $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$:

- (1) $(a + b)^* = a^* + b^*$
- (2) $(ab)^* = b^*a^*$
- (3) $(\alpha a)^* = \bar{\alpha}a^*$
- (4) $(a^*)^* = a$
- (5) $|a^*a| = |a|^2$

Fact 1.2. \mathbb{C} is a C^* -algebra under complex conjugation.

Definition 1.3.

- Let S be a linear operator from H into H . The operator S is called *bounded* if the set $\{\|Su\| : u \in H, \|u\| = 1\}$ is bounded in \mathbb{C} . If A is bounded we define the *norm* of A by:

$$\|S\| = \sup_{u \in H, \|u\|=1} \|Su\|$$

Let H be a Hilbert space. We denote by $B(H)$ the algebra of all bounded linear operators from H to H .

- Given a linear operator $S : H \rightarrow H$, its *adjoint operator*, denoted S^* is the unique linear operator $S^* : H \rightarrow H$ such that for every $u, v \in H$, $\langle Su|v \rangle = \langle u|S^*v \rangle$.

Remark 1.4. The unicity of the adjoint comes from a duality relation between H and H' . See [11], Volume 1, Chapter VI, Section 2.

Fact 1.5. $B(H)$ is a C^* -algebra under the adjoint operation.

Remark 1.6. There are three important topologies on $B(H)$: The norm topology, the *strong* and the *weak*. Strong topology is the topology of pointwise convergence. In weak topology $T_k \rightarrow T$ if for all v and $w \in H$, $\langle T_k v | w \rangle \rightarrow \langle T v | w \rangle$

Definition 1.7. Let X be a locally compact Hausdorff space. We set:

$$\mathcal{C}_0(X, \mathbb{C}) = \{f \in \mathcal{C}(X, \mathbb{C}) \mid \text{for all } \epsilon > 0 \text{ there exist } K \subseteq X \text{ compact such that } |f(x)| < \epsilon \text{ for all } x \in X \setminus K\}.$$

Fact 1.8. If X is a topological space (resp. locally compact Hausdorff space), $\mathcal{C}(X, \mathbb{C})$ (resp. $\mathcal{C}_0(X, \mathbb{C})$) is a C^* -algebra under the adjoint operation on complex functions.

Definition 1.9. Let \mathcal{A} be a complex Banach algebra with identity. Let $a \in \mathcal{A}$. Then the *spectrum* of a is the set $\sigma_{\mathcal{A}}(a)$ of complex values $\lambda \in \mathbb{C}$ such that the element $a - \lambda 1$ is not invertible. The set $\mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ is called the *resolvent set* for a and is denoted $\rho_{\mathcal{A}}(a)$. Let S a bounded linear operator on a Hilbert space H . Then $\sigma(S)$ denotes the *spectrum* of S as an element of the algebra $B(H)$ i.e. the set of complex values t such that $S - tI$ is not an invertible operator in $B(H)$; also $\rho(a)$ is the resolvent set of S as an element of the algebra $B(H)$ i.e. the set of complex values t such that $S - tI$ is an invertible operator in $B(H)$.

Definition 1.10. Let $a \in \mathcal{A}$. Then, a is called *selfadjoint* if $a = a^*$. If in addition \mathcal{A} has an identity and $\sigma_{\mathcal{A}}(a) \subseteq [0, \infty)$, then a is called *positive*. The set of positive elements in \mathcal{A} is denoted by \mathcal{A}_+ .

Fact 1.11. A self-adjoint operator S is positive if and only if $\langle Su|u \rangle \geq 0$ for all $u \in H$.

Notation 1.12. Let $a, b \in \mathcal{A}$. By $a \leq b$ we mean that $b - a$ is positive.

Definition 1.13. A self-adjoint element $a \in \mathcal{A}$ a *square root* of a positive self-adjoint operator $b \in \mathcal{A}$ if $a^*a = a^2 = b$.

Fact 1.14 (Theorem VI.9 in [11] Vol 1). Every positive operator on H admits a unique positive square root.

Definition 1.15.

- Let \mathcal{A} be a C^* -algebra. A *unit* in \mathcal{A} is an element $e \in \mathcal{A}$ such that $ea = ae = a$ for all $a \in \mathcal{A}$.
- A C^* -algebra with a unit is called *unital*.
- Let \mathcal{A} be a C^* -algebra. An *approximate unit* in \mathcal{A} is a net $(a_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{A}$ such that $\lambda < \mu$ implies that $a_\lambda \leq a_\mu$ and $\lim \|x - xa_\lambda\| = 0$ for all $x \in \mathcal{A}$.

Fact 1.16 (Theorem 1.4.2 in [10]). Each C^* -algebra contains an approximate unit.

Definition 1.17. Let \mathcal{A} be a $*$ -algebra with a unity. An element $a \in \mathcal{A}$ is called *normal* if $aa^* = a^*a$.

Notation 1.18. Let N be a normal operator on a Hilbert space H . Then C^* -subalgebra of $B(H)$ generated by N and I is denoted by $C^*(N)$.

Definition 1.19. Let \mathcal{A} an unital C^* -algebra and let $a \in \mathcal{A}$ an invertible element. a is called *unitary* if $a^{-1} = a^*$.

Fact 1.20. Let S be a linear operator on H . Then, S is unitary if and only if S is an isometry.

Definition 1.21. Let S a bounded linear operator from H to H . Then,

- S is called of *finite rank* if $\dim(SH) < \infty$.
- S is called *compact* if the image under S of the unit ball is relatively compact.

Fact 1.22 (Theorem VI.13 in [11]). A bounded linear operator K is compact if and only if there exists a sequence of finite rank operators F_n such that F_n converges to K in the norm topology.

Notation 1.23. The set of all compact operators in H is denoted by \mathcal{K} .

Definition 1.24.

- Let \mathcal{A} be a complex Banach algebra. A subalgebra $\mathcal{I} \subseteq \mathcal{A}$ is called an *ideal* of \mathcal{A} , if for every $a \in \mathcal{A}$ and $i \in \mathcal{I}$, ia and ai are in \mathcal{I} .
- Let \mathcal{I} be an ideal in \mathcal{A} . We define an equivalence relation in \mathcal{A} in the following way:

$$a \simeq_{\mathcal{I}} b \text{ if and only if } b - a \in \mathcal{I}$$

- Let \mathcal{I} be an ideal in \mathcal{A} . The *quotient algebra* \mathcal{A}/\mathcal{I} is the set of equivalence classes according $\simeq_{\mathcal{I}}$.

Fact 1.25. Let \mathcal{I} be an ideal in a C^* -algebra \mathcal{A} . Then \mathcal{A}/\mathcal{I} is a C^* -algebra.

Fact 1.26. The set $\mathcal{K}(H)$ of all the compact operators on H defines a closed ideal in $B(H)$

Definition 1.27. The algebra $B(H)/\mathcal{K}(H)$ is called the *Calkin algebra* on H . If N is a normal operator on H , the C^* -subalgebra of $B(H)$ generated by equivalence class of N and the equivalence class of I is denoted $\tilde{C}^*(N)$.

1.2. Functional calculus for bounded operators on H . In this subsection we study the functional calculus for a bounded normal operator on H . This is will be imitated in last section although in much more complicated ways. Reference used here is [3]

Definition 1.28. Let $N \in B(H)$ be a normal operator on H . Let $\hat{\cdot}$ be the function that for a polynomial $p(x) = \sum_{i,j=0}^n c_{ij} z^i \bar{z}^j$ assigns $\hat{p}(N) = \sum_{i,j=0}^n c_{ij} N^i (N^*)^j$

Fact 1.29 (Theorem 2.3.1 in [3]). Let $N \in B(H)$ be a normal operator on H . Then the previous function extends uniquely to an isometric C^* -algebra isomorphism of $\mathcal{C}(\sigma(N), \mathbb{C})$ onto $C^*(N)$.

Fact 1.30 (Theorem 2.6.3 in [3]). Let X be a compact metrizable space. Let $\pi \in \text{rep}(\mathcal{C}(X, \mathbb{C}), H)$ be such that H is the closed linear span of $\pi(A)H$. Then, π extends uniquely to a representation $\tilde{\pi} \in \text{rep}(\mathcal{B}(X, \mathbb{C}), H)$.

Corolary 1.31. Let N be a normal operator on H , let $\mathcal{B}(\sigma(N))$ the algebra of borel functions from $\sigma(N)$ into the complex numbers, and let $B(H)$ the algebra of linear operators on H . Then there exist a isometric monomorphism $\pi : \mathcal{B}(\sigma(N)) \rightarrow B(H)$ such that $\pi(\bar{f}) = (\pi(f))^*$, $\pi(1) = I$ and if $f = \sum_i \sum_j a_{ij} z^i \bar{z}^j$, then $\pi(f) = \sum_i \sum_j a_{ij} N^i (N^*)^j$, where by 1 we denote the constant function on $\sigma(N)$ with value 1.

2. IMPORTANT THEOREMS ABOUT C^* ALGEBRAS

In this section we give some important facts about C^* -algebras like Von Neumann's bicommutant theorem, Kaplanski's density theorem, the up down theorem and Gelfand Theorem. The only source here is [10].

2.1. Von Neumann bicommutant theorem.

Definition 2.1. Given a subset $M \subseteq B(H)$, we define de *commutant* M' of M the set,

$$M' = \{S \in B(H) \mid \forall T \in M, ST = TS\}$$

Theorem 2.2 (Von Neumann Bicommutant Theorem. Theorem 2.2.2 in [10]). Let M be a subalgebra of $B(H)$ containing the identity. Then the following are equivalent:

- (1) $M = M''$.
- (2) M is weakly closed.
- (3) M is Strongly closed.

Definition 2.3. A C^* -subalgebra of $B(H)$ satisfying any of this equivalent condition is called a *Von Neumann algebra*.

2.2. Kaplanski Density Theorem.

Theorem 2.4 (Kaplanski density theorem. Theorem 2.3.3. in [10]). Let \mathcal{A} be a C^* -subalgebra of $B(H)$ with strong closure M . Then the unit ball \mathcal{A}^1 of \mathcal{A} is strongly dense in the unit ball M^1 of M . Furthermore, \mathcal{A}_{sa}^1 is strongly dense in M_{sa}^1 , \mathcal{A}_+^1 is strongly dense in M_+^1 and the unitary group of \mathcal{A} is dense in the unitary group of M .

2.3. Up-Down Theorem.

Definition 2.5. Let $M \subseteq B(H)$. Then by M^m we denote the set of elements of $B(H)$ which are strong limits of increasing nets of elements of M .

Theorem 2.6 (Up-Down Theorem. Theorem 2.4.4. in [10]). *Let M be a C^* -subalgebra of $B(H)$. Then, M is a Von Neumann algebra if and only if $(M_{sa}^m = M_{sa})$.*

2.4. Gelfand Theorem and abelian C^* -algebras. The main goal of this subsection is Theorem 2.11 which states that Gelfand transform on an abelian C^* -algebra is a $*$ -algebra isomorphism. This will be used in next subsection to relate radon measures (regular locally finite) measures with positive bounded linear functionals over \mathcal{A} which will be defined in Subsection 3.2

Definition 2.7. Let \mathcal{A} be an abelian C^* -algebra. The set:

$$Sp(\mathcal{A}) = \{\omega : \mathcal{A} \rightarrow \mathbb{C} \mid \omega \text{ is a } C^*\text{-algebra homomorphism}\},$$

is called the *spectrum* or the *maximal ideal space* of \mathcal{A} . This set is considered to have the weak* topology inherited from \mathcal{A}'

Theorem 2.8 (Corollary I.2.6 in [4]). *If \mathcal{A} is abelian, then $Sp(\mathcal{A})$ is a locally compact Hausdorff space. If in addition \mathcal{A} has a unit, $Sp(\mathcal{A})$ is compact.*

Corolary 2.9. *Let $\mathcal{A} = C_0(K)$ where K is a locally compact Hausdorff space. Then $Sp(\mathcal{A}) = K$*

Definition 2.10. Let \mathcal{A} be an abelian C^* -algebra. Then the morphism Γ from \mathcal{A} to $\mathcal{C}_0(Sp(\mathcal{A}), \mathbb{C})$ given by:

$$\begin{aligned} \Gamma(a) : Sp(\mathcal{A}) &\rightarrow \mathbb{C} \\ f &\rightarrow \Gamma(a)(f) = f(a) \end{aligned}$$

for every $a \in \mathcal{A}$ is called the *Gelfand transform*.

Theorem 2.11 (Theorem I.3.1. in [4]). *Let \mathcal{A} be an abelian C^* -algebra. Then the Gelfand transform is an isometric $*$ -isomorphism.*

Theorem 2.12 (Proposition 1.1.8. and Remark 1.1.4 in [10]). *Let \mathcal{A} be abelian C^* -algebra with unity. Let $a \in \mathcal{A}$ be a normal element, i.e. $aa^* = a^*a$. Let \mathcal{B} be the smallest C^* -subalgebra of \mathcal{A} containing a and 1. Then \mathcal{B} is isometrically $*$ -isomorphic to $\mathcal{C}_0(\sigma_A(a), \mathbb{C})$ and $\sigma_A(a) = \sigma_B(a)$.*

3. IMPORTANT CONCEPTS ABOUT C^* ALGEBRAS

Here, some important concepts like normal morphism, central cover and Gelfand-Naimark-Construction are discussed. Once again the only source here is [10].

3.1. Bounded positive linear functionals and Radon measures. This a very important subsection of preliminaries. The main goal is to understand positive linear functionals over \mathcal{A} and relate them with Radon measures over $Sp(\mathcal{A})$. That is exactly Theorem 3.10. It is important to mention that this measures will correspond to types of vectors in \mathcal{H} , and the measure theoretic relations between them will correspond to model theoretic relations between the corresponding types.

Definition 3.1. Let \mathcal{A}' be the dual space of \mathcal{A} . An element $\phi \in \mathcal{A}'$ is called *positive* if $\phi(a) \geq 0$ whenever $a \in \mathcal{A}$ is positive. Set of positive functionals is denoted by \mathcal{A}'_+ .

Fact 3.2. Let \mathcal{A} be an abelian C^* -algebra of operators on a Hilbert space H , and let $v \in \mathcal{H}$. Then the function ϕ_v on \mathcal{A} such that for every $S \in \mathcal{A}$, $\phi_v(T) = \langle Sv | v \rangle$ is a positive linear functional.

Proof. Linearity is clear. Let S be a positive selfadjoint operator in \mathcal{A} , let Q be its square root and let $v \in H$. Then $\langle Sv | v \rangle = \langle Q^*Qv | v \rangle = \langle Qv | Qv \rangle \geq 0$ \square

Remark 3.3. It is easy to prove that \mathcal{A}' is a C^* -algebra: if $\phi \in \mathcal{A}'$ we can define ϕ^* by $\phi^*(a) = \overline{\phi(a^*)}$ for all $a \in \mathcal{A}$.

Definition 3.4. A positive linear functional ϕ on \mathcal{A} is called a *state* if $\|\phi\| = 1$. The set of the states on \mathcal{A} is denoted by $S(\mathcal{A})$ or if there is no confusion, simply S .

Definition 3.5. Let ϕ and ψ positive linear functionals on \mathcal{A} . They are called *orthogonal* ($\phi \perp \psi$) if $\|\phi - \psi\| = \|\phi\| + \|\psi\|$. Also, ϕ is called *dominated* by ψ ($\phi \leq \psi$) if the functional $\psi - \phi$ is positive.

Recall the following definitions:

Definition 3.6. Let X be a topological space. Let μ and ν be two complex measures on the borel sets of X . μ is said to be *absolutely continuous* ($\mu \ll \nu$) with respect to ν if for every borel set $V \subseteq X$, $\nu(V) = 0$ implies $\mu(V) = 0$. μ and ν are said *mutually singular* ($\mu \perp \nu$) if $\text{supp}(\mu) \cap \text{supp}(\nu) = \emptyset$.

Theorem 3.7 (Lebesgue decomposition theorem. Theorem 6.10 a) in [12]). *Let μ and ν be two borel positive measures on a topological space X . Then there exists borel positive measures ν_1 and ν_2 such that $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$.*

Theorem 3.8 (Radon-Nikodim theorem. Theorem 6.10 6) in[12]). *Let μ and ν be two borel positive measures on a topological space X such that $\nu \ll \mu$. Then there exists $f \in L^1(X, \mu)$ such that, for every Borel set V , $\nu(V) = \int_V f d\mu$.*

Theorem 3.9 (Riesz representation theorem. Theorem 6.19 in [12]). *Let X be a locally compact Hausdorff space. Then any bounded linear functional Φ on $\mathcal{C}_0(X, \mathbb{C})$ is represented by a single complex borel regular measure in the sense that:*

$$\Phi f = \int_X f d\mu,$$

for every $f \in \mathcal{C}_0(X, \mathbb{C})$. Moreover, $\|\phi\| = |\mu|(X)$.

Theorem 3.10. *Let X be a locally compact Hausdorff space. The Radon (locally finite and inner regular) measures on X are in correspondence with the positive linear functionals on $\mathcal{C}_0(X, \mathbb{C})$ and this correspondence is isometric.*

Proof. Let μ be a positive Radon measure on X . Let ϕ_μ the functional on $\mathcal{C}_0(X, \mathbb{C})$ given by $\phi_\mu(f) = \int_X f d\mu$. Then ϕ_μ is positive. Conversely, By the Riesz representation theorem any positive linear functional ϕ on $\mathcal{C}_0(X, \mathbb{C})$ determines a positive measure μ_ϕ . The Riesz representation theorem guarantees that the correspondence is isometric too. \square

Lemma 3.11. $\mu \ll \nu$ if and only if there is a positive real number M such that $\phi_\mu \leq M\phi_\nu$.

Proof. Let $\phi \leq \psi$ and V such that $\mu_\psi(V) = 0$. Then we have that $\mu_\psi = \mu_\phi + \mu_{\psi-\phi}$. Since both μ_ϕ and $\mu_{\psi-\phi}$ are positive measures, $0 = \mu_{\psi-\phi}(V) = \mu_\phi(V)$ and $\mu_\phi \ll \mu_\psi$. Assume now that $\mu \ll \nu$. By Radon-Nikodim theorem, there exists $f \in$

$L^1(X, \nu)$ positive such that $d\mu = fd\nu$. Let $M = \int_X fd\nu$ then $d(\nu - \mu) = (M - f)d\nu$ defines a positive Radon measure corresponding to the functional $M\psi - \phi$. \square

Lemma 3.12. $\mu \perp \nu$ if and only $\phi_\mu \perp \phi_\nu$.

Proof. Suppose $\mu \perp \nu$. Then $\|\phi_\mu - \phi_\nu\| = |\mu - \nu|(X) = |\mu(X)| + |\nu(X)| = \|\phi_\mu\| + \|\phi_\nu\|$. Conversely, suppose that $\phi \perp \psi$. Take $A \subseteq \text{supp}(\phi)$ and $B \subseteq \text{supp}(\psi)$. Then $\mu_\phi(B) = \phi(\chi_B) = 0$ and $\mu_\psi(A) = \psi(\chi_A) = 0$. \square

From now on, let \mathcal{A} be an abelian C^* -algebra of operators in a Hilbert space H , and let \mathcal{M} be its strong closure in $B(H)$

Theorem 3.13. *Given $v \in H$, the positive linear functional ϕ_v defined in Fact 3.2 defines Radon measure over $Sp(\mathcal{A})$. This measure will be called the spectral measure defined by v and will be denoted μ_v*

Fact 3.14 (Generalized Luzin's theorem, Theorem 2.7.3 in [10]). For each $b \in \mathcal{M}$, each projection $p_0 \in \mathcal{M}$ and each set $\{v_1, \dots, v_n\} \subseteq H$, there is a projection $p \in \mathcal{M}$ with $p \leq p_0$, $\|(p_0 - p)v_i\| < \epsilon$ for all i , and an element $a \in \mathcal{A}$ such that $\|a\| \leq \|bP_0\| + \epsilon$ and $bp = ap$.

Corolary 3.15. *Let $v \in \mathcal{H}$, $b \in \mathcal{M}$. Then there exist a sequence $(a_k) \subseteq \mathcal{A}$ such that $a_kv \rightarrow bv$ when $k \rightarrow \infty$.*

Proof. Take p_k a sequence of projections coconverging to identity and apply Luzin's theorem. \square

3.2. Representations of C^* -algebras. This is the last subsection of preliminaries. It deals with the representations of a C^* -algebra. The main theorem here is Theorem 3.20 which gives a canonical way to build representations of a C^* -algebra called the *Gelfand-Naimark-Segal* construction. This concept will be very helpful in defining definable closures and forking between types.

Definition 3.16. Let \mathcal{A} be a C^* -algebra. A *representation* is an algebra homomorphism $\pi : \mathcal{A} \rightarrow B(H)$ such that for all $a \in \mathcal{A}$, $\pi(a^*) = (\pi(a))^*$. In this case H is called an \mathcal{A} -*module*. A Hilbert subspace $H' \subseteq H$ is called an \mathcal{A} -*submodule* of H if H' is closed under π . H is called \mathcal{A} -*irreducible* if H has no proper non trivial \mathcal{A} -submodules. The set of representations of an algebra \mathcal{A} is denoted $rep(\mathcal{A}, B(H))$.

Definition 3.17. Let G be a group of bounded operators on H . H is called a G -*module*. A Hilbert subspace $H' \subseteq H$ is called a G -*submodule* of H if H' is closed under the action of G . H is called G -*irreducible* if H has no proper non trivial G -submodules.

Fact 3.18. Let G be the group of all the unitary operators on H . Then H is an irreducible G -module.

Proof. Given $v, w \in H$ such that $\|v\| = \|w\|$, there exists a unitary operator U such that $Uv = w$. \square

Definition 3.19.

- Let (π, H) a representation of a C^* -algebra \mathcal{A} . (π, H) is called *non-degenerate* if for every nonzero vector $v \in H$, there exists $a \in \mathcal{A}$ such that $\pi(a)v \neq 0$. (π, H) is called *cyclic* if there exists a vector v_π such that $\pi(\mathcal{A})v_\pi$ is dense in H . Such a vector is called a *cyclic vector* for the representation (π, H) .

- Let (p_i, H_i) for $i \in I$ a family of representations of \mathcal{A} . We define a representation $\oplus \pi_i$ on $\oplus H_i$ on this way: Let $v = \sum_i v_i$ and $a \in \mathcal{A}$, $\oplus \pi_i(a)v = \sum_i \pi_i(a)v_i$.

Theorem 3.20 (Theorem 3.3.3. and Remarks 3.4.1. in [10]). *Let ϕ be a positive functional on \mathcal{A} . Then there exists a cyclic representation (π_ϕ, H_ϕ) with a cyclic vector $v_\phi \in H_\phi$ tal que for all $a \in \mathcal{A}$, $\phi(a) = \langle \pi_\phi(a)v_\phi | v_\phi \rangle$. This representation is called the Gelfand-Naimark-Segal construction. In the case when \mathcal{A} is abelian and has a unit, $H_\phi = L^2(Sp(\mathcal{A}), \mu_{v_\phi})$, μ_{v_ϕ} is the radon measure corresponding to ϕ , v_ϕ is the identity function and π_ϕ is the morphism that takes $\mathcal{C}_0(Sp(\mathcal{A}), \mathbb{C})$ into the multiplication operators on $L^2(Sp(\mathcal{A}), \mu_{v_\phi})$.*

Definition 3.21. Two representations (π_1, H_1) and (π_2, H_2) are said *unitarily equivalent* if there exists an isometry U from H_1 to H_2 such that for every $a \in \mathcal{A}$, $U\pi_1(a)U^* = \pi_2(a)$.

Theorem 3.22 (Proposition 3.3.7 in [10]). *Two cyclic representations (π_1, H_1) and (π_2, H_2) are unitarily equivalent by mean of an isometry U such that $Uv_{\pi_1} = v_{\pi_2}$ if and only if for all $a \in \mathcal{A}$, $\langle \pi_1(a)v_{\pi_1} | v_{\pi_1} \rangle = \langle \pi_2(a)v_{\pi_2} | v_{\pi_2} \rangle$*

Definition 3.23. Two representations (π_1, H_1) and (π_2, H_2) are said *aproximately unitarily equivalent* if there exists a sequence of unitary operators U_n from H_1 to H_2 such that for every $a \in \mathcal{A}$, and every $x, y \in H_2$ $\langle U_n \pi_1(a) U_n^* x | y \rangle = \langle \lim \pi_2(a) | y \rangle$.

Theorem 3.24 (Theorem II.5.8 in [4]). *Two representations (π_1, H_1) and (π_2, H_2) are aproximately unitarily equivalent if and only if for all $a \in \mathcal{A}$, $\text{rank}(\pi_1(a)) = \text{rank}(\pi_2)$*

Definition 3.25. We define the *discrete part* of \mathcal{A} , denoted by \mathcal{A}_d , the norm closure of the set of $a \in \mathcal{A}$ such that $\text{rank}(a)$ is finite.

Fact 3.26. \mathcal{A}_d is the ideal of compact operators in \mathcal{A} .

Definition 3.27.

- Let \mathcal{A} be an abelian C^* -subalgebra of $B(H)$. The *discrete spectrum* of \mathcal{A} ($Sp_d(\mathcal{A})$) is the spectrum of the algebra \mathcal{A}_d .
- We define the *essential part* of \mathcal{A} , denoted by \mathcal{A}_e , the algebra $\mathcal{A}/\mathcal{A}_d$. The *essential spectrum* of \mathcal{A} is the spectrum of the algebra \mathcal{A}_e .

Remark 3.28 (Generalized Weyl's criterion for essential spectrum). From previous definition it follows that if $t \in Sp(\mathcal{A})$, then $t \in Sp_e(\mathcal{A})$ if and only if, for every open neighborhood $V \subseteq Sp(\mathcal{A})$ of t , $\dim(\mathcal{H}_V) = \infty$

Lemma 3.29. *Suppose $\mathcal{A} = C_0(Sp(\mathcal{A}))$. Let $f \in \mathcal{A}$ and let $V \subseteq Sp(\mathcal{A})$ an open set contained $\text{supp}(f)$. Then $\dim(\mathcal{H}_V) \leq \text{rank}(f)$.*

Proof. Let $f \in \mathcal{A}$. $f \upharpoonright H_V : H_V \rightarrow H$ has trivial kernel because $\text{supp}(f) \subseteq \text{supp}(\mu_v)$ for every $v \in H_V$. \square

Theorem 3.30. *Two representations (π_1, H_1) and (π_2, H_2) of an abelian C^* -algebra \mathcal{A} are aproximately unitarily equivalent if and only if the following holds:*

- (1) $Sp(\pi_1(\mathcal{A})) = Sp(\pi_2(\mathcal{A}))$
- (2) $Sp_e(\pi_1(\mathcal{A})) = Sp_e(\pi_2(\mathcal{A}))$

(3) Given $t \in Sp(\mathcal{A})$, the dimension of the space of eigenvectors for t in H_1 is equal to the corresponding dimension in H_2

Proof. Immediate consequence from previous Lemma. \square

Theorem 3.31 (Corollary 3.3.8. in [10]). *Let ϕ and ψ be positive functionals on \mathcal{A} . If $\phi \leq \alpha\psi$ for some $\alpha \in \mathbb{C}$ then (π_ϕ, H_ϕ) is unitarily equivalent to a subrepresentation of (π_ψ, H_ψ) .*

Definition 3.32.

- A subset $F \subseteq S(\mathcal{A})$ is called *separating* if for every $a \in \mathcal{A}$ $\phi(a) = 0$ for every $\phi \in F$ implies that $a = 0$, i.e. the set $\{\phi\}$ is separating.
- Let $\phi \in S(\mathcal{A})$. ϕ is said to be *faithful* if for every $a \in \mathcal{A}_+$, $\phi(a) = 0$ implies that $a = 0$. A *faithful representation* is a representation (π, H) such that if $\pi(a) = 0$ then $a = 0$ for $a \in \mathcal{A}_+$.

Notation 3.33. For each $\phi \in S$, let $(\pi_\phi, H_\phi, v_\phi)$ be the Gelfand-Naimark-Segal construction of ϕ . For $F \subseteq S$ let $H_F = \bigoplus_{\phi \in F} H_\phi$ and $\pi_F = \bigoplus_{\phi \in F} \pi_\phi$.

Theorem 3.34 (Proposition 3.7.4 in [10]). *If $F \subseteq S$ is separating, then (π_F, H_F) is a faithful representation.*

Definition 3.35. The representation (π_S, H_S) is called the *universal representation*

Definition 3.36. A von Neumann M is called *σ -finite* each set of pairwise orthogonal non-zero projections in M is countable. A projection $p \in M$ is called *σ -finite* if pMp is σ -finite

3.3. Normal morphisms.

Definition 3.37. Let M and N von Neumann subalgebras of $B(H)$ and $B(K)$ respectively. A positive linear map $\rho : M \rightarrow N$ is said to be *normal* if for each bounded monotone increasing net $(x_\lambda)_{\lambda \in L}$ in M_{sa} with limit x , the net $(\rho(x_\lambda))_{\lambda \in L}$ increases to $\rho(x)$ in N_{sa} .

Theorem 3.38.

Remark page 28 in [10]: *Every normal morphism is continuous.*

Proposition 2.5.2 in [10]: *Any isomorphism between von Neumann algebras is normal.*

Theorem 2.5.3 in [10]: *Let ρ be a normal morphism between von Neumann algebras M and N . Then, $\ker(\rho)$ is strongly closed in M and $\rho(M)$ is strongly closed in N .*

Corollary 2.5.5 in [10]: *The image of a von Neumann algebra under a normal morphism is isomorphic to Mq for some projection $q \in M \cap M'$.*

3.4. The central cover.

Definition 3.39.

- (1) The set $M \cap M'$ is called the *center* of M and is denoted by $Z(M)$.
- (2) A von Neumann algebra is said to be a *factor* if $Z(M) \sim \mathbb{C}$.

Definition 3.40. Let $x \in M_{sa}$. The *central cover* of x is the infimum of all $z \geq x$ in $Z(M)_{sa}$. This central cover is denoted by $c(x)$.

4. DECOMPOSITION OF ALGEBRAS

The aim of this section is to understand the ways to decompose a C^* algebra. This decomposition theorems tell us that there exists a classification theory for C^* algebras and therefore, from the model theoretical point of view, it is not such a chaotical theory. There are two initially independent ways to decompose a C^* algebra: By means of direct integrals of factors and by means of type I, II and III components. This two ways can be combined into a single and powerfull decomposition concept that can be very helpful in analysing a particular representation of a C^* -algebra as an algebra of operators over some Hilbert space H . The only source for this section is [10].

4.1. Factor decomposition.

Definition 4.1. Let \mathcal{A} be a C^* -algebra. The set of equivalence clases in the set of irreducible representations of \mathcal{A} under the relation of unitary equivalence, is called the *spectrum* \mathcal{A} and is denoted by $Sp(\mathcal{A})$. This agrees with the definition of spectrum of an abelian C^* -algebra.

Definition 4.2.

- (1) An ideal $I \subseteq \mathcal{A}$ is called *primitive* if whenever $xIy \subseteq I$, then either $x \in I$ or $y \in I$.
- (2) Let \mathcal{A} be a C^* -algebra. The set of primitive ideals in \mathcal{A} is called the *primitive spectrum* of \mathcal{A} , and is denoted by $Sp(\mathcal{A})$.

Theorem 4.3 (Theorem 4.4.2 in [10]). *For each $x \in \mathcal{A}_{sa}''$ there is a bounded real function \tilde{x} on $Sp(\mathcal{A})$ such that $\pi''(c(x)) = \tilde{x}(t)1$ when (π, H) is an irreducible representation of \mathcal{A} belonging to t .*

Definition 4.4. Let \mathcal{A} be a C^* -subalgebra $B(H)$.

- (1) \mathcal{A} is said to be a *Borel**-algebra if \mathcal{A}_{sa} is *sequentially closed*, i.e., it contains the strong limits of all increasing or decreasing nets inside it.
- (2) The least Borel*-algebra containing \mathcal{A} is called the *enveloping Borel**-algebra of \mathcal{A} and is denoted by \mathcal{B} . The centre of \mathcal{B} is denoted by \mathcal{C} .
- (3) A subset $F \subseteq \mathcal{A}$ is called *D-Borel* if there exists an element $x \in \mathcal{C}$ such that \tilde{x} is the characteristic function of F .
- (4) The *D-Borel structure* of $Sp(\mathcal{A})$ is the σ -algebra of D-Borel sets in $Sp(\mathcal{A})$.
- (5) Let ϕ be a state of \mathcal{A} . Then the measure μ_ϕ defined by ϕ on the D-Borel structure of $Sp(\mathcal{A})$ given by $\mu_\phi(F) = \phi(\tilde{x}_F)$ is called the *central measure* associated with ϕ .

Definition 4.5. Let \mathcal{A} be a separable c^* -algebra.

- (1) A nondegenerate representation (π, H) of \mathcal{A} is called a *factor representation* if $\pi(\mathcal{A})''$ is a factor.
- (2) A state ϕ is called *factorial* if its Gelfand-Naimark-Segal construction (π_ϕ, H_ϕ) is a factor representation.

- (3) We denote by $F(\mathcal{A})$ the set of all factorial states of \mathcal{A} , and by $Sp_f(\mathcal{A})$ the set of equivalence classes of factor representations of \mathcal{A} . $Sp_f(\mathcal{A})$ is called the *factor spectrum* of \mathcal{A} .

Definition 4.6.

- (1) A set of vectors in a Hilbert space is called *total* if the only vector orthogonal to that set is the zero vector.
- (2) Let T a borel space and $\{H_t \mid t \in T\}$ a family of separable Hilbert spaces. A *vector field* is an element of $\eta \in \mathbb{P}_T H_t$.
- (3) A Borel field is a family of Hilbert spaces $(H_t)_{t \in T}$ if there exists a sequence of vector fields $(\eta_n) \subseteq \mathbb{P}_T H_t$ such that:
 - (a) For each $t \in T$, the set $\{\eta_n(t) \mid n \in \mathbb{N}\}$ is total in H_t .
 - (b) For every $n, m \in \mathbb{N}$, the function $t \rightarrow \langle \eta_n(t) \mid \eta_m(t) \rangle$.
- (4) An *orthonormal base* for a Borel field $(H_t)_{t \in T}$ is a sequence of vector fields such that previous conditions hold and:
 - (a) For $t \in T$, the set $\{\eta_n(t) \mid n < \dim(H_t) + 1\}$ is an orthonormal basis of H_t .
 - (b) $\eta_n(t) = 0$ if $n > \dim(H_t)$.
- (5) If ξ_1 and ξ_2 are vector fields in a borel field H_t , their inner product is defined by:

$$\langle \xi_1(t) \mid \xi_2(t) \rangle = \sum_n \langle \xi_1 \mid \eta_n(t) \rangle \langle \eta_n(t) \mid \xi_2(t) \rangle$$

- (6) Let μ be a borel measure on T and H_t borel field. A vector field ξ on T is said to be *square integrable* if the function $t \rightarrow \|\xi(t)\|^2$ is integrable with respect to μ .
- (7) A vector field $\xi(t)$ is called *null* if the function $t \rightarrow \|\xi(t)\|^2$ is null with respect to μ .
- (8) The set of all vector fields over H_t which are square integrable with respect to μ modulo the null vector fields is called the *direct integral* of the borel field $(H_t)_{t \in T}$, and is denoted by:

$$\int_T^\oplus H_t d\mu$$

- (9) If $\xi_1, \xi_2 \in \int_T^\oplus H_t d\mu$ their *inner product* is defined by:

$$\langle \xi_1 \mid \xi_2 \rangle = \int_T \langle \xi_1(t) \mid \xi_2(t) \rangle d\mu$$

Theorem 4.7 (Theorem 4.12.4 in [10]). *Let (π, H) be a non degenerate separable representation of a separable C^* -algebra \mathcal{A} . Then there exists:*

- (1) A central D -Borel measure μ on $\sigma_f(\mathcal{A})$ corresponding to some state on \mathcal{A} ;
- (2) A borel field $(H_t) \mid t \in \sigma_f(\mathcal{A})$ of Hilbert spaces;
- (3) An isometry u from $\int_T^\oplus H_t d\mu$ onto H .

4.2. Type I, II and III von Neumann algebras.

Definition 4.8. Let \mathcal{A} be a C^* -algebra.

- (1) A *weight* is a function $\phi : \mathcal{A}_+ \rightarrow [0, \infty]$ such that:
 - (a) $\phi(\alpha x) = \alpha \phi(x)$ for all $x \in \mathcal{A}_+$ and $\alpha \in \mathbb{R}_+$;
 - (b) $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in \mathcal{A}_+$.

- (2) A weight ϕ on \mathcal{A} is said to be *densely defined* if the set

$$\mathcal{A}_+^\phi = \{x \in \mathcal{A}_+ \mid \phi(x) < \infty\}$$

is dense in \mathcal{A}_+ .

- (3) If \mathcal{A} is a von Neumann algebra, \mathcal{A} is said to be *semifinite* if \mathcal{A}_+^ϕ is weakly dense in \mathcal{A} .
- (4) If \mathcal{A} is a Borel* algebra, \mathcal{A} is said to be σ -finite if \mathcal{A}_+^ϕ contains an increasing sequence with limit 1.
- (5) A weight ϕ is said to be *lower semicontinuous* if for every $\alpha \in \mathbb{R}_+$, the set $\{x \in \mathcal{A}_+ \mid \phi(x) \leq \alpha\}$ is closed.
- (6) If \mathcal{A} is a Borel*-algebra, a weight ϕ on \mathcal{A} is called σ -normal if there exists a sequence ϕ_n of sequentially normal positive functionals such that $\phi = \sum_n \phi_n$.
- (7) A weight which is σ -finite and σ -normal is called a σ -weight.
- (8) A *trace* on \mathcal{A} is a weight on \mathcal{A} such that $\phi(uxu^*) = \phi(x)$ for all $x \in \mathcal{A}_+$ and $u \in \mathcal{A}$ unitary.
- (9) If \mathcal{A} is a von Neumann algebra, then:
- (a) \mathcal{A} is said to be *finite* if there is a faithful, normal, finite trace on \mathcal{A} .
 - (b) \mathcal{A} is said to be *semifinite* if there is a faithful, normal, semifinite trace on \mathcal{A} .
 - (c) \mathcal{A} is said to be *properly infinite* if there is no nonzero normal, finite trace on \mathcal{A} .
 - (d) \mathcal{A} is said to be *purely infinite* if there is no nonzero normal, semifinite trace on \mathcal{A} .

Theorem 4.9 (Proposition 5.4.2 in [10]). *Each von neumann algebra M has a unique decomposition $M = M_1 \oplus M_2 \oplus M_3$ such that M_1 is finite, M_2 is semifinite but properly infinite and M_3 is purely infinite.*

Definition 4.10. Let M be a von Neumann algebra.

- (1) A projection $p \in M$ is said to be *abelian* if pMp is abelian.
- (2) M is said to be of *type I* if there exists a abelian projection $p \in M$ such that $c(p) = 1$.
- (3) M is said to be of *type II* if it is semifinite but doesnot contain any nonzero abelian projection.
- (4) M is said to be of *type III* if it is purely infinite.

Corolary 4.11. *Each von neumann algebra M has a unique decomposition $M = M_1 \oplus M_2 \oplus M_3$ such that M_1 is of type I, M_2 is of type II and M_3 is of type III.*

Definition 4.12. Let M be a von Neumann algebra. If M is of type I it is said to be of *type I_n* , where $n = 1, \dots, \infty$, if there are n disjoint, equivalent, abelian projections p_1, \dots, p_n in M such that $p_1 + p_2 + \dots + p_n = 1$.

Theorem 4.13 (Proposition 5.5.7 in [10]). *Each von Neuman algebra of type I on a separable Hilbert space H has a unique decomposition $M = \prod_{n=1}^{\infty} M_n$ where each M_n is of type I_n .*

Definition 4.14. Let M be a type II von Neuman algebra. Then

- (1) M is called of *type II_1* if M is finite.
- (2) M is called of *type II_∞* if M is properly infinite.

Theorem 4.15 (Proposition 5.5.13 in [10]). *Each von Neuman algebra on a separable Hilbert space has a unique decomposition:*

$$M = M_1 \oplus M_2 \oplus \cdots, M_\infty \oplus M_{II_1} \oplus M_{II_\infty} \oplus M_{III},$$

where the M_n are of type I_n , M_{II_1} is of type II_1 , M_{II_∞} is of type II_∞ and M_{III} is of type III .

Theorem 4.16 (Theorem 5.8.9 in [10]). *Let M be a von Neumann algebra on a separable Hilbert space and let $\int_T^\oplus H_t d\mu$ as in 4.7. Let \mathcal{A} be one the symbols I_n , II_1 , II_∞ and III . Then M is of type α if and only if μ -almost all the M_t 's are of type α .*

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