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Hilbert spaces expanded with a unitary operator

Camilo Argoty*^{1,2} and Alexander Berenstein**¹

¹ Universidad de los Andes, Departamento de Matemáticas, Carrera 1 N 18A-10, Bogotá, Colombia

² Universidad Sergio Arboleda, Departamento de Matemáticas

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We study Hilbert spaces expanded with a unitary operator with a countable spectrum. We show that the theory of such a structure is ω -stable and admits quantifier elimination.

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1 Introduction

This paper deals with the expansion of a Hilbert space with a unitary operator with a countable spectrum from the perspective of continuous logic (for an introduction to this subject see [4]). We study the model-theoretic properties of such a structure in terms of the spectrum of the operator. In other words, our goal here is to study the relation between Model Theory and Spectral Theory when the spectrum of the operator involved is countable. The main results are the following theorems.

Theorem 1.1 *Let U be a unitary operator with pure point spectrum in a Hilbert space \mathcal{H} . Then the theory of the structure $(\mathcal{H}, U) = (H, +, 0, \langle \cdot, \cdot \rangle, U)$ admits quantifier elimination, is separably categorical (see Definition 3.18), is ω -stable (see Definition 3.20), and the projection into every eigenspace is definable.*

Theorem 1.2 *Let U be a unitary operator whose spectrum has countably many non-empty accumulation points. Then the theory of the structure $(\mathcal{H}, U) = (H, +, 0, \langle \cdot, \cdot \rangle, U)$ admits quantifier elimination and is ω -stable. The projection into an eigenspace is definable if and only if the corresponding eigenvalue is an isolated point in the spectrum.*

The theory of the structure $(H, +, 0, \langle \cdot, \cdot \rangle, U)$, where U is a unitary operator, was first studied by Henson and Iovino in [11], where they observed that it is stable. A geometric characterization of forking in such structures was first done by Berenstein and Buechler [7] after adding to the structure the projections corresponding to the Spectral Decomposition Theorem. Ben Yaacov, Usvyatsov, and Zadka characterized the unitary operators which correspond to generic automorphisms of a Hilbert space as those unitary transformations whose spectrum is the unit circle S^1 . In this work we study the definable aspects of the spectral decomposition, we classify the separable models of the expansions, and we study orthogonality of types when U has a countable spectrum. We also give a different proof of the characterization of forking given in [7]: we provide an explicit freeness relation and prove some of its properties which show that it coincides with non-forking, without adding to the structure the projections corresponding to the spectral decomposition.

The framework for this work is continuous logic, we assume the reader is familiar with notions such as definability, definable and algebraic closure, and forking. The background can be found in [4, 5]. This paper is divided as follows. In the second section we give a brief introduction to Spectral Theory. In the third section we study the expansions of a Hilbert space with unitary operators with a pure point (finite) spectrum and with a countably infinite spectrum.

* e-mail: c-argoty@uniandes.edu.co

** Corresponding author: e-mail: aberenst@uniandes.edu.co

2 Preliminaries: Spectral Theory

The following section is based on [1] and [12]. Let $\mathcal{H} = (H, +, 0, \langle, \rangle)$ be an infinite-dimensional Hilbert space over \mathbb{C} .

Definition 2.1 Let A be a linear operator from \mathcal{H} into \mathcal{H} . The operator A is called *bounded* if the set

$$\{\|Au\| : u \in H, \|u\| = 1\}$$

is bounded in \mathbb{R} . If A is bounded, we define *the norm of A* by

$$\|A\| = \sup_{u \in H, \|u\|=1} \|Au\|.$$

Definition 2.2 A sequence of linear operators $\{A_n\}_{n \in \omega}$ converges uniformly to an operator A if

$$\lim_{n \rightarrow \infty} \|A - A_n\| = 0.$$

Definition 2.3 Given a linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$, its *adjoint operator*, denoted by A^* , is the linear operator $A^* : \mathcal{H} \rightarrow \mathcal{H}$ such that for every $u, v \in \mathcal{H}$, $\langle Au, v \rangle = \langle u, A^*v \rangle$.

Definition 2.4 A linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is called *self-adjoint* if $A = A^*$.

Definition 2.5 Let A be a linear operator from \mathcal{H} to \mathcal{H} . A is called *normal* if A commutes with its adjoint A^* .

Definition 2.6 Given a normal operator A , a complex number λ is called an *eigenvalue* or *punctual spectral value* of A if the operator $A - \lambda I$ is not one-to-one. A complex number λ is called a *continuous spectral value* if the operator $A - \lambda I$ is one-to-one and the operator $(A - \lambda I)^{-1}$ is densely defined but is unbounded.

Fact 2.7 [1, Theorem 1, Section 93] *A point r belongs to the continuous spectrum of an operator A if and only if there is an orthonormal sequence of elements $(f_n : n \in \omega)$ such that $\lim_{n \rightarrow \infty} (Af_n - rf_n) = 0$.*

Definition 2.8 *The spectrum of an operator A , denoted by $\sigma(A)$, is the set of the punctual and continuous spectral values.*

It is well-known that if A is a bounded normal operator, then $\sigma(A)$ is a bounded subset of \mathbb{C} , and that if A is self-adjoint, then $\sigma(A)$ is a subset of \mathbb{R} .

Definition 2.9 A self-adjoint operator A different from the zero operator is said to be *positive*, written $A \geq 0$, if $\langle Au, u \rangle \geq 0$ for all $u \in \mathcal{H}$. If A and B are self-adjoint operators, we write $A \geq B$ if $A - B$ is positive.

Proposition 2.10 *For a self-adjoint operator A , A^2 is positive.*

Proof. Clear. □

Definition 2.11 A self-adjoint operator Q is a *square root of a positive self-adjoint operator A* if $Q^2 = A$.

Fact 2.12 [12, Theorem 1, Section 36] *Let A be a self-adjoint operator and let E_+ be the projection of \mathcal{H} onto the null-space of the operator $A - Q$, where Q is the positive square root of A^2 . Then the following hold:*

1. Any bounded operator R that commutes with A commutes with E_+ .
2. $AE_+ \geq 0$, $A(I - E_+) \leq 0$.

Fact 2.13 [12, Theorem 2, Section 36] *Let A be a bounded self-adjoint operator; let r be a real number, and let $E_+(r)$ be the projection operator constructed for $A_r = A - rI$ according to Fact 2.12. If we denote by E_r the projection operator $I - E_+(r)$, then the family $\{E_r\}_{r \in \mathbb{R}}$ satisfies the following conditions:*

1. Any bounded operator R that commutes with A commutes with E_r .
2. $E_r \leq E_s$ if $r < s$.
3. E_r is continuous on the left: $\bigcup_{s < r} E_s = E_r$.
4. $E_r = 0$ for $-\infty < r \leq m$ and $E_r = I$ for $M < r < \infty$, where $m = \inf(\sigma(A))$ and $M = \sup(\sigma(A))$.

Definition 2.14 The family $\{E_r\}_{r \in [m, M]}$, where $\sigma(A) \subseteq [m, M]$, is called *the resolution of the identity generated by A* .

Fact 2.15 [12, Spectral Decomposition Theorem, Section 35] *Let A be a bounded self-adjoint operator. Then for every $\varepsilon > 0$, $A = \int_m^{M+\varepsilon} r \, dE_r$, where the integral is to be interpreted as limit of finite sums in the sense of uniform convergence in the space of operators, and $m = \inf(\sigma(A))$ and $M = \sup(\sigma(A))$.*

Let A be as above. The Spectral Decomposition Theorem can be interpreted in the following way: given $\varepsilon > 0$ and $\delta > 0$ there is N such that for any partition

$$m = r_0 < s_0 = r_1 < s_1 = r_2 < \dots < s_N = M + \varepsilon$$

with $\max\{s_k - r_k\} < 2(M + \varepsilon)/N$, if we write $E(\Delta_k)$ for $E_{s_k} - E_{r_k}$, we have $\|A - \sum_{k=1}^N r_k E(\Delta_k)\| < \delta$.

Definition 2.16 Let A be a normal operator. *The real and imaginary parts of A are the operators*

$$A_r = \frac{1}{2}(A + A^*) \quad \text{and} \quad A_i = \frac{1}{2}(A - A^*).$$

Fact 2.17 *The real and imaginary parts A_r and A_i of a normal operator A are self-adjoint.*

Proof. For $u, v \in H$,

$$\langle A_r u, v \rangle = \langle \frac{1}{2}(A + A^*)u, v \rangle = \langle u, \frac{1}{2}(A^* + A^{**})v \rangle = \langle u, \frac{1}{2}(A + A^*)v \rangle = \langle u, A_r v \rangle.$$

Similarly for A_i . □

Corollary 2.18 *If A is a normal operator, then A_r and A_i have integral representations*

$$A_r = \int_a^b s \, dE_s \quad \text{and} \quad A_i = \int_a^b t \, dF_t,$$

where $E_s F_t = F_t E_s$, and we can write $A = \int \int (s + it) dE_s dF_t$.

Definition 2.19 A normal operator U on \mathcal{H} is said to be *unitary* if U is a bijection and $\langle Uu, Uv \rangle = \langle u, v \rangle$ for any $u, v \in H$.

Fact 2.20 [1, Section 71] *Let U be a unitary operator. Then there exists a unique resolution of the identity $\{E_r : r \in [0, 2\pi]\}$ such that $U^k = \int_0^{2\pi} e^{ikt} \, dE_t$ for all $k \in \mathbb{Z}$.*

We want to know if we can recover the resolution of the identity $\{E_r : r \in \sigma(A)\}$ from A .

Theorem 2.21 [2, Theorem 2.6.3] *Let N be a normal operator on \mathcal{H} , let $\mathcal{B}(\sigma(N))$ be the algebra of Borel functions from $\sigma(N)$ into the complex numbers, and let $B(\mathcal{H})$ be the algebra of linear operators on \mathcal{H} . Then there exists an isometric monomorphism $\pi : \mathcal{B}(\sigma(N)) \rightarrow B(\mathcal{H})$ such that $\pi(\bar{f}) = (\pi(f))^*$, $\pi(1) = I$, and if $f = \sum_i \sum_j a_{ij} z^i \bar{z}^j$, then $\pi(f) = \sum_i \sum_j a_{ij} N^i (N^*)^j$, where by 1 we denote the constant function on $\sigma(N)$ with value 1.*

Fact 2.22 *Let A be a self-adjoint operator on \mathcal{H} . Let $m = \inf\{\sigma(A)\}$ and $M = \sup(\sigma(A))$, and let f be the function on $[m, M]$ such that $f(x) = 0$ if $x \leq 0$ and $f(x) = 1$ if $x > 0$. Then f is approximated in measure by polynomials.*

Proof. This function clearly belongs to $L^2[m, M]$ and the polynomials are dense in $L^2[m, M]$, so the conclusion follows. □

Theorem 2.23 *Let A be a self-adjoint operator, and let E_+ be the projection of \mathcal{H} onto the null-space of the operator $A - Q$, where Q is the positive square root of A^2 . Then E_+ is approximated by operators having the form $g(A)$, where g is a polynomial.*

Proof. For any $v \in H$ we have

$$E_+(v) = \int_m^M g(r) dE_r(v),$$

where g is the function defined in Fact 2.22. But by Fact 2.22, $g(x)$ can be approximated in measure by a sequence of polynomials $P_k(x)$. This implies that E_+ can also be approximated punctually by the sequence $P_k(A)$ of polynomials in A , this is $P_k(A)(v)$ converges to $E_+(v)$ for every v . □

It is important to note that for a self-adjoint operator A on a Hilbert space \mathcal{H} , E_+ is the limit of a sequence of polynomials in A , but the sequence does not converge uniformly. That is, given a unit vector $v \in H$, $E_+(v)$ can be approximated by polynomials in $A(v)$, but the rate of convergence depends on v .

Definition 2.24 *The essential spectrum $\sigma_e(U)$ of a linear operator U is the set of accumulation points of the spectrum of U together with the set of eigenvalues of infinite multiplicity. The finite spectrum $\sigma_{\text{fin}}(U)$ of a linear operator U , the complement of $\sigma_e(U)$, is the set of isolated points of the spectrum of U of finite multiplicity.*

3 Unitary operators

We begin this section by characterizing the theory of the expansion of a Hilbert space with a unitary operator in terms of the spectrum of the operator using an instance of a result by C. Ward Henson.

Theorem 3.1 *Let A and B be two bounded normal operators on Hilbert spaces \mathcal{H}_A , \mathcal{H}_B , respectively. Then the structures $(H_A, +, 0, \langle, \rangle, A)$ and $(H_B, +, 0, \langle, \rangle, B)$ are elementarily equivalent if and only if*

1. $\sigma_e(A) = \sigma_e(B)$,
2. $\dim\{x \in H_A : Ax = \lambda x\} = \dim\{x \in H_B : Bx = \lambda x\}$ for $\lambda \in \mathbb{C} \setminus \sigma_e(A)$.

We will provide a proof for the result when the operators involved have a countable spectrum and are unitary. Henson's original proof (unpublished), based on a theorem by Voiculescu (see [10]), yields stronger information than our proof, in particular, it shows that the theory $\text{Th}(\mathcal{H}, A)$ is separably categorical up to perturbations of the operator.

We follow the standard way of dealing with Hilbert spaces, we write the structure as $(H, +, 0, \langle, \rangle)$, and we work inside the unit ball. We denote by L the language of Hilbert spaces and by L_U the language of Hilbert spaces with a new unary function, which we denote as U . Whenever we quantify over H we mean we are quantifying over its unit ball.

Notation 3.2 For $\lambda \in \sigma(U)$, we denote by H_λ the set $\{x \in H : Ux = \lambda x\}$, by P_λ the projection operator onto the space H_λ .

Theorem 3.3 *Let U_A and U_B be two unitary operators with countable spectrum on a Hilbert space \mathcal{H} . Then the structures $(H_A, +, 0, \langle, \rangle, U_A)$ and $(H_B, +, 0, \langle, \rangle, U_B)$ are elementarily equivalent if and only if*

1. $\sigma_e(U_A) = \sigma_e(U_B)$,
2. $\dim\{x \in H_A : U_A x = \lambda x\} = \dim\{x \in H_B : U_B x = \lambda x\}$ for $\lambda \in S^1 \setminus \sigma_e(U_A)$.

Proof.

(\Rightarrow) We will show that the conditions $\lambda \in \sigma_e(U_A)$ and $\dim\{x \in H_A : U_A x = \lambda x\} = m$ can be expressed as a set of statements in continuous logic. We may assume the structures are separable and we will write H instead of H_A, H_B . Assume that the structures $(H, +, 0, \langle, \rangle, U_A)$ and $(H, +, 0, \langle, \rangle, U_B)$ are elementarily equivalent. Let σ^A be the spectrum of U_A and let σ^B be the spectrum of U_B . For every $\mu \in S^1 \setminus \sigma^A$, let $\eta = d(\mu, \sigma^A)$. Then the statement $\sup_{u \in H} (\eta \|u\| - \|Uu - \mu u\|) = 0$ is true for $(H, +, 0, \langle, \rangle, U_A)$ and thus true for $(H, +, 0, \langle, \rangle, U_B)$. Therefore $\sigma^B \subseteq \sigma^A$. In a similar way we can show that $\sigma^A \subseteq \sigma^B$. For every $\lambda \in \sigma_{\text{fin}}^A$ whose eigenspace is of dimension m_λ , we have that the statement

$$\inf_{u_1, u_2, \dots, u_{m_\lambda}} \max_{i \neq j} (|\langle u_i, u_j \rangle|, \|u_i\| - 1, |Uu_i - \lambda u_i|) = 0$$

is valid in $(H, +, 0, \langle, \rangle, U_A)$. Thus the same statement is true for $(H, +, 0, \langle, \rangle, U_B)$. From this condition and the fact that λ is an isolated point in the spectrum it easily follows that

$$\dim\{x \in H : U_A x = \lambda x\} \leq \dim\{x \in H : U_B x = \lambda x\}.$$

In a similar way we can prove that

$$\dim\{x \in H : U_A x = \lambda x\} \geq \dim\{x \in H : U_B x = \lambda x\}.$$

Also, for every $\lambda \in \sigma_e^A$ and every $k \geq 1$,

$$\inf_{u_1, u_2, \dots, u_k} \max_{i \neq j} (|\langle u_i, u_j \rangle|, \|u_i\| - 1, |Uu_i - \lambda u_i|) = 0$$

is true for $(H, +, 0, \langle, \rangle, U_A)$ and thus also true for $(H, +, 0, \langle, \rangle, U_B)$. This implies that $\sigma_e^A = \sigma_e^B$.

(\Leftarrow) Conversely, assume that

$$\sigma_e^A = \sigma_e^B \quad \text{and} \quad \dim\{x \in H : U_A x = \lambda x\} = \dim\{x \in H : U_B x = \lambda x\} \text{ for } \lambda \in \sigma_{\text{fin}}^A.$$

Let $\sigma_e = \sigma_e^A$. We will construct a common elementary extension of both structures (\mathcal{H}_A, U_A) and (\mathcal{H}_B, U_B) . We may assume, exchanging (\mathcal{H}_A, U_A) and (\mathcal{H}_B, U_B) for elementary superstructures, (\mathcal{H}_A, U_A) and (\mathcal{H}_B, U_B) to be \aleph_0 -saturated. Let

$$H_A^{\text{fin}} = \bigoplus_{\lambda \in \sigma_{\text{fin}}} H_A^\lambda, \quad \text{where } H_A^\lambda = \{x \in H_A : U_A x = \lambda x\}.$$

Note that H_A^λ is the domain of a finite-dimensional Hilbert subspace \mathcal{H}_A^λ of \mathcal{H}_A . Let

$$H_B^{\text{fin}} = \bigoplus_{\lambda \in \sigma_{\text{fin}}} H_B^\lambda, \quad \text{where } H_B^\lambda = \{x \in H_B : U_B x = \lambda x\},$$

then $(\mathcal{H}_B^{\text{fin}}, U_B)$ is a substructure of (\mathcal{H}_B, U_B) . Then for each $\lambda \in \sigma_{\text{fin}}$, $(\mathcal{H}_A^\lambda, U_A) \cong (\mathcal{H}_B^\lambda, U_B)$, so

$$(\mathcal{H}_A^{\text{fin}}, U_A) \cong (\mathcal{H}_B^{\text{fin}}, U_B).$$

We denote this common substructure as \mathcal{H}_{fin} .

Let $\mathcal{H}_C = \mathcal{H}_A \oplus_{\mathcal{H}_{\text{fin}}} \mathcal{H}_B$, the free amalgamation of \mathcal{H}_A and \mathcal{H}_B over \mathcal{H}_{fin} , and let U_C be the induced unitary map on \mathcal{H}_C determined by U_A and U_B . We will prove that $(\mathcal{H}_A, U_A) \preceq (\mathcal{H}_C, U_C)$ and $(\mathcal{H}_B, U_B) \preceq (\mathcal{H}_C, U_C)$. From this we get that the structures $(H_A, +, 0, \langle, \rangle, U_A)$ and $(H_B, +, 0, \langle, \rangle, U_B)$ are elementarily equivalent.

Claim 3.3.1 $(\mathcal{H}_A, U_A) \preceq (\mathcal{H}_C, U_C)$.

We use the Tarski-Vaught test. Let $\varphi(x_1, \dots, x_n, y)$ be a formula and let $\bar{a} = (a_1, \dots, a_n) \in H_A^n$. Assume that $\inf\{\varphi^{\mathcal{H}_C, U_C}(a_1, \dots, a_n, b) : b \in H_C\} = r$ and let $\varepsilon > 0$. Let $b \in H_C$ be such that

$$\varphi(a_1, \dots, a_n, b) - r < \varepsilon.$$

We may write $b = b_A + d$, where $b_A = P_{H_A}(b)$, so $d \perp H_A$.

There exist $k \geq 1$ and an L -formula $\psi(x_{11}, \dots, x_{n1}, y_1; \dots; x_{1k}, \dots, x_{nk}, y_k)$ such that

$$\begin{aligned} (\mathcal{H}_C, U_C) \models \sup_{x_1} \dots \sup_{x_n} \sup_y |\varphi(x_1, \dots, x_n, y) \\ - \psi(x_1, \dots, x_n, y; \dots; U^k(x_1), \dots, U^k(x_k), U^k(y))| = 0. \end{aligned}$$

It is clear that $d \in \bigoplus_{\lambda \in \sigma_e} H_\lambda^B$, so $d = \sum_{\lambda \in \sigma_e} P_\lambda(d)$. Note that $P_\lambda(d) \perp H_A$, so $P_\lambda(d) \perp P_\lambda(a_i)$ for each $i \leq n$. Since (\mathcal{H}_A, U_A) is \aleph_0 -saturated, $\dim(H_A^\lambda) = \infty$ for each $\lambda \in \sigma_e$, so there exists $c_\lambda \in H_A^\lambda$ such that

$$\|c_\lambda\| = \|P_\lambda(d)\| \quad \text{and} \quad c_\lambda \perp P_\lambda(a_1), \dots, P_\lambda(a_n), P_\lambda(b_A).$$

Let $c = \sum_{\lambda \in \sigma_e} c_\lambda$ and let $b' = b_A + c$. Then

$$\text{tp}(c, \dots, U^k(c)/\{\bar{a}, U(\bar{a}), \dots, U^k(\bar{a}), b_A\}) = \text{tp}(d, \dots, U^k(d)/\{\bar{a}, U(\bar{a}), \dots, U^k(\bar{a}), b_A\}),$$

so $\text{tp}(b, \dots, U^k(b)/\{\bar{a}, U(\bar{a}), \dots, U^k(\bar{a})\}) = \text{tp}(b', \dots, U^k(b')/\{\bar{a}, U(\bar{a}), \dots, U^k(\bar{a})\})$, so

$$\psi(a_1, \dots, a_n, b, \dots, U^k(a_1), \dots, U^k(a_k), U^k(b)) = \psi(a_1, \dots, a_n, b', \dots, U^k(a_1), \dots, U^k(a_k), U^k(b'))$$

and

$$\varphi(a_1, \dots, a_n, b') - r < \varepsilon.$$

Since ε was arbitrary we get

$$\inf\{\varphi^{\mathcal{H}_C, U_C}(a_1, \dots, a_n, b) : b \in H_C\} = r = \inf\{\varphi^{\mathcal{H}_C, U_C}(a_1, \dots, a_n, b') : b' \in H_A\}.$$

Thus the Tarski-Vaught test holds and $(\mathcal{H}_A, A) \preceq (\mathcal{H}_C, C)$, as we wanted. \square

The previous theorem is proved in [6] when $\sigma^A = \sigma_e^A = S^1$.

Let U be a unitary operator in a separable Hilbert space \mathcal{H} and σ be the corresponding spectrum. The previous result says that the theory of the structure $(\mathcal{H}, U) = (H, +, 0, \langle, \rangle, U)$ determines the dimension of the eigenspace corresponding to the isolated points of the spectrum. We strengthen this observation by showing the definability of these eigenspaces.

Notation 3.4 For $\lambda \in \sigma$, we denote by B_λ the unit ball in the space \mathcal{H}_λ .

Recall that a closed set $D \subseteq H$ is definable if the function $\text{dist}(x, D)$ is a definable predicate. We will use the following characterization of definable sets and functions.

Fact 3.5 [4, Proposition 9.18] *Let $D \subseteq H$ be closed. Then D is definable in (\mathcal{H}, U) if and only if there is a definable predicate $P : H^n \rightarrow [0, 1]$ such that*

$$P(x) = 0 \text{ for all } x \in D \quad \text{and} \quad \forall \varepsilon \exists \delta (\forall x \in H^n) (P(x) \leq \delta \Rightarrow d(x, D) \leq \varepsilon).$$

Fact 3.6 [4, Proposition 9.22] *Let (\mathcal{H}', U') be κ -saturated, where κ is uncountable. Let $A \subseteq H'$ have cardinality $< \kappa$. Let $f : H'^n \rightarrow H'$ be any function, and let \mathcal{G}_f be its graph. Then the following are equivalent:*

1. f is definable in (\mathcal{H}', U') over A .
2. \mathcal{G}_f is type-definable in (\mathcal{H}', U') over A .

Lemma 3.7 *Let $\lambda \in \sigma$ be isolated, let $\chi = d(\lambda, \sigma \setminus \{\lambda\})$, and let $u \in H$. If $\|Uu - \lambda u\| < \varepsilon$, then*

$$\|u - P_\lambda(u)\| < \frac{\varepsilon}{\chi}.$$

Proof. Let $\lambda' \in \sigma$ be isolated and let $u \in H$. Then

$$Uu - \lambda' u = \sum_{\lambda \in \sigma, \lambda \neq \lambda'} (\lambda - \lambda') P_\lambda(u)$$

and

$$\|Uu - \lambda' u\|^2 = \sum_{\lambda \in \sigma, \lambda \neq \lambda'} |\lambda - \lambda'|^2 \cdot \|P_\lambda(u)\|^2.$$

We have $|\lambda - \lambda'|^2 \geq \chi^2$ for all $\lambda \in \sigma$ such that $\lambda \neq \lambda'$. If $\|Uu - \lambda' u\|^2 < \varepsilon^2$, $\chi^2 \sum_{\lambda \in \sigma, \lambda \neq \lambda'} \|P_\lambda(u)\|^2 < \varepsilon^2$, so that $\|u - P_{\lambda'}(u)\|^2 = \|\sum_{\lambda \in \sigma, \lambda \neq \lambda'} P_\lambda(u)\|^2 < \frac{\varepsilon^2}{\chi^2}$. \square

Theorem 3.8 *For every isolated $\lambda \in \sigma$ the subspace H_λ is definable.*

Proof. By Lemma 3.7, we have that if $\|Uu - \lambda u\| \rightarrow 0$, then $d(u, H_\lambda) \rightarrow 0$ uniformly on \mathcal{H} . By Fact 3.5, \mathcal{H}_λ is definable. \square

For general $\lambda \in \sigma$, we have a weaker result.

Theorem 3.9 *For every $\lambda \in \sigma$ the subspace \mathcal{H}_λ is a zero-set.*

Proof. H_λ is the set of solutions of the statement $\|Uv - \lambda v\| = 0$. \square

We will later show (Corollary 3.27) that for $\lambda \in \sigma$ non-isolated, \mathcal{H}_λ is not definable.

Theorem 3.10 *For every isolated λ the projection P_λ is definable.*

Proof. We work in $(\mathcal{H}', U') \succeq (\mathcal{H}, U)$, an \aleph_1 -saturated elementary extension of (\mathcal{H}, U) . By Fact 3.6, it suffices to show that \mathcal{G}_{P_λ} is type-definable. Consider the formula

$$\varphi(u, v) = \|Uv - \lambda v\|^2 + \sup_{x \in B_\lambda} |\langle v - u, x \rangle|^2.$$

Then $\varphi(u, v) = 0$ if and only if $v = P_\lambda(u)$. \square

Finally we use Theorem 3.3 to give an explicit axiomatization of the theory of the structure $(H, +, 0, \langle, \rangle, U)$.

We write $\sigma = \sigma_{\text{fin}} \cup \sigma_e$, and for each $\lambda \in \sigma_{\text{fin}}$ let m_λ be the dimension of \mathcal{H}_λ . Let T_σ be the theory of Hilbert spaces together with the following statements:

1. $\sup_u \sup_v |\langle u, v \rangle - \langle Uu, Uv \rangle| = 0$.
2. $\sup_u \inf_v \|u - Uv\| = 0$.
3. (Schema)
 - (a) For $n \in \mathbb{N}$ and $\lambda \in \sigma_e$, $\inf_{u_1, u_2, \dots, u_n} \max_{i \neq j} (|\langle u_i, u_j \rangle|, \|u_i\| - 1, |Uu_i - \lambda u_i|) = 0$.
 - (b) For $\lambda \in \sigma_{\text{fin}}$, $\chi = d(\lambda, \sigma \setminus \{\lambda\})$, and $0 < \varepsilon < 1$,

$$\inf_{u_1, u_2, \dots, u_{m_\lambda}} \sup_v \max_{i \neq j} (|\langle u_i, u_j \rangle|, \|u_i\| - 1, |Uu_i - \lambda u_i|, \varepsilon \div \|Uv - \lambda v\|, \|v - \sum_{k=1}^m \langle v, u_i \rangle u_i\| \div \frac{\varepsilon}{\chi}) = 0.$$

4. (Schema) For $\mu \in S^1 \setminus \sigma$ and $\eta = d(\mu, \sigma)$, $\sup_u (\eta \|u\| \div \|Uu - \mu u\|) = 0$.

Proposition 3.11 *The theory T_σ axiomatizes $\text{Th}(H, +, 0, \langle, \rangle, U)$.*

Proof. Clearly the statements 1., 2., 3.(a), 4. hold for $(H, +, 0, \langle, \rangle, U)$. By Lemma 3.7 the statements 3.(b) are true for $(H, +, 0, \langle, \rangle, U)$. Now assume $(H, +, 0, \langle, \rangle, A)$ satisfies the axioms above. The first two axioms say that A is an inner product preserving linear maps from \mathcal{H} onto \mathcal{H} and thus is a unitary operator. The fourth axiom says that the spectrum σ^A of A is contained in σ . Finally by the third axiom,

$$\sigma_e^A = \sigma_e \quad \text{and} \quad \dim\{x \in H : Ax = \lambda x\} = \dim\{x \in H : Ux = \lambda x\} \text{ for } \lambda \in \sigma^A \setminus \sigma_e^A.$$

By Theorem 3.3 we get $\text{Th}(H, +, 0, \langle, \rangle, U) = \text{Th}(H, +, 0, \langle, \rangle, A)$. □

It is important to note that U^* is quantifier-free definable in terms of U . Let (\mathcal{H}, U) be an \aleph_1 -saturated model of T_σ and let

$$\begin{aligned} P(x, y) &= \|U^*x - y\|^2 \\ &= \langle U^*x - y, U^*x - y \rangle \\ &= \|U^*x\|^2 - \langle U^*x, y \rangle - \langle y, U^*x \rangle + \|y\|^2 \\ &= \|Ux\|^2 - \langle x, Uy \rangle - \langle Uy, x \rangle + \|y\|^2. \end{aligned}$$

If $P(x, y) \rightarrow 0$, then $d(U^*x, y) \rightarrow 0$.

Our next goal is to study the separable models of $\text{Th}(H, +, 0, \langle, \rangle, U)$ and to understand the space of types over \emptyset . This analysis depends heavily on the properties of the spectrum σ of U and we divide our work accordingly.

3.1 Finite spectrum

Let U be a unitary operator with a pure point spectrum. In this subsection we will study the model theory of the structure $(H, 0, +, \langle, \rangle, U)$, characterizing the definable closure, the algebraic closure, and properties of the space of types. We write σ for the spectrum of U .

Theorem 3.12 *All the projections P_λ , $\lambda \in \sigma$, can be expressed as polynomials in U .*

Proof. Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$. Then $u = \sum_{k=1}^n P_{\lambda_k}(u)$ for each $u \in H$ and $U^m(u) = \sum_{k=1}^n \lambda_k^m P_{\lambda_k}(u)$ for each $m \in \mathbb{N}$. So

$$\begin{pmatrix} u \\ U(u) \\ \vdots \\ U^{n-1}(u) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} \begin{pmatrix} P_{\lambda_1}(u) \\ P_{\lambda_2}(u) \\ \vdots \\ P_{\lambda_n}(u) \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

is the Vandermonde matrix, whose determinant is $(-1)^n \prod_{i < j} (\lambda_i - \lambda_j) \neq 0$. This implies

$$\begin{pmatrix} P_{\lambda_1}(u) \\ P_{\lambda_2}(u) \\ \vdots \\ P_{\lambda_n}(u) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} u \\ U(u) \\ \vdots \\ U^{n-1}(u) \end{pmatrix}. \quad \square$$

Corollary 3.13 For each $\lambda \in \sigma$, the projection P_λ is definable.

Proof. Clear. □

Notice that the previous corollary was already known by Theorem 3.10.

Theorem 3.14 The structure $(H, 0, +, \langle, \rangle, U)$ admits quantifier elimination. Furthermore, the type $\text{tp}(\bar{a})$ is for $\bar{a} = (a_1, \dots, a_n) \in H^n$ determined by the values of $\langle P_\lambda(a_k), P_\lambda(a_l) \rangle$ for $\lambda \in \sigma$ and $k, l \leq n$.

Proof. Let $\bar{a} = (a_1, \dots, a_n), \bar{b} = (b_1, \dots, b_n) \in H^n$. Let us write $\text{qftp}(a_1, \dots, a_n)$ for the set of quantifier-free statements valid for $a_1, \dots, a_n \in H$. We will show that $\text{qftp}(a_1, \dots, a_n)$ determines $\text{tp}(a_1, \dots, a_n)$. Assume that $\text{qftp}(\bar{a}) = \text{qftp}(\bar{b})$. Then, by Theorem 3.12,

$$\langle P_\lambda(a_k), P_\lambda(a_l) \rangle = \langle P_\lambda(b_k), P_\lambda(b_l) \rangle$$

for $\lambda \in \sigma$ and $k, l = 1, \dots, n$. Let $\lambda \in \sigma$. We can assume without loss of generality that there exists $0 < k \leq n$ such that the set $\{P_\lambda(a_1), \dots, P_\lambda(a_k)\}$ is linearly independent and the elements of $\{P_\lambda(a_{k+1}), \dots, P_\lambda(a_n)\}$ can be expressed as linear combinations of the elements of $\{P_\lambda(a_1), \dots, P_\lambda(a_k)\}$. Because $\text{qftp}(\bar{a}) = \text{qftp}(\bar{b})$ we have that the set $\{P_\lambda(b_1), \dots, P_\lambda(b_k)\}$ is linearly independent and the elements of $\{P_\lambda(b_{k+1}), \dots, P_\lambda(b_n)\}$ can be expressed in terms of the elements of $\{P_\lambda(b_1), \dots, P_\lambda(b_k)\}$ with exactly the same linear combinations as before.

For each $\lambda \in \sigma$, let B_1^λ be an orthonormal basis of $H_\lambda \cap \{P_\lambda(a_1), \dots, P_\lambda(a_k)\}^\perp$ and let B_2^λ be an orthonormal basis of $H_\lambda \cap \{P_\lambda(b_1), \dots, P_\lambda(b_n)\}^\perp$. Then

$$\text{qftp}(P_\lambda(a_1), \dots, P_\lambda(a_n), B_1^\lambda) = \text{qftp}(P_\lambda(b_1), \dots, P_\lambda(b_n), B_2^\lambda).$$

For each $\lambda \in \sigma$, let $f_\lambda : \mathcal{H}_\lambda \rightarrow \mathcal{H}_\lambda$ be the linear transformation generated by the map that sends $P_\lambda(a_k)$ into $P_\lambda(b_k)$ for $k = 1, \dots, n$ and is a bijection between B_1^λ and B_2^λ .

Recall that

$$\mathcal{H} = \bigoplus_{\lambda \in \sigma} \mathcal{H}_\lambda.$$

Let $f : \mathcal{H} \rightarrow \mathcal{H}$ be the automorphism induced by the family $(f_\lambda : \lambda \in \sigma)$. (We can write $u \in H$ as $\sum_{\lambda \in \sigma} u_\lambda$, where $u_\lambda = P_\lambda(u)$. Then $f(\sum_{\lambda \in \sigma} u_\lambda) = \sum_{\lambda \in \sigma} f_\lambda(u_\lambda)$.) It is clear that $f(\bar{a}) = \bar{b}$ and $f(U(c)) = U(f(c))$ for any $c \in H$, so f is an automorphism of the structure (\mathcal{H}, U) . This implies $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$. □

Lemma 3.15 Let $A \subseteq H$. Then $\text{dcl}(A)$ is the closed Hilbert space generated by $\{P_\lambda(a) : a \in A, \lambda \in \sigma\}$.

Proof. Let A be a subset of H and E be the closed vector space generated by $\{P_\lambda(a) : a \in A, \lambda \in \sigma\}$.

We first show that $E \subseteq \text{dcl}(A)$. Let $b \in E$. Then there exist a sequence $(c_{n,\lambda} : n \in \omega, \lambda \in \sigma)$ of complex numbers and a sequence $(a_{n,\lambda} : n \in \omega, \lambda \in \sigma)$ of elements of A such that $b = \sum_{k=0, \lambda \in \sigma}^{\infty} c_{k,\lambda} P_{\lambda}(a_{k,\lambda})$. Thus

$$(\forall \varepsilon > 0) \exists N (\forall n \geq N) (\|b - \sum_{k=0, \lambda \in \sigma}^n c_{k,\lambda} P_{\lambda}(a_{k,\lambda})\| < \varepsilon).$$

Let $R(x) = \|x - b\|$ and $\varphi_n(x) = \|x - \sum_{k=0, \lambda \in \sigma}^n c_{k,\lambda} P_{\lambda}(a_{k,\lambda})\|$. Then

$$(\forall \varepsilon > 0) \exists N (\forall n \geq N) \forall x (|R(x) - \varphi_n(x)| < \varepsilon)$$

and $\{b\}$ is definable over A .

Let $b \notin E$, then for some $\lambda_0 \in \sigma$, $P_{\lambda_0}(b) \notin E$. Since there are infinitely many vectors in $H_{\lambda_0} \cap E^{\perp}$ with norm $\|P_{\lambda_0}(b - P_E(b))\|$, there are infinitely many realizations of $\text{tp}(b/A)$ and therefore $b \notin \text{dcl}(A)$. \square

Lemma 3.16 *Let $A \subseteq H$. Then $\text{acl}(A)$ is the closed Hilbert space generated by the union of $\text{dcl}(A)$ with all the finite-dimensional subspaces \mathcal{H}_{λ} with $\lambda \in \sigma_{\text{fin}}$.*

Proof. Every ball B_{λ} for $\lambda \in \sigma_{\text{fin}}$ is algebraic over \emptyset because the closed unit ball of a finite-dimensional space is compact. Therefore $\text{dcl}(A) \cup \bigcup_{\lambda \in \sigma_{\text{fin}}} H_{\lambda} \subseteq \text{acl}(A)$. Conversely, let E be the closure of the space generated by $\text{dcl}(A)$ and $\bigcup_{\lambda \in \sigma_{\text{fin}}} H_{\lambda}$. If $b \notin E$, then $P_{\lambda_0}(b) - P_{\lambda_0}(P_E(b)) \neq 0$ for some $\lambda_0 \in \sigma_e$. Without loss of generality the dimension of $H_{\lambda_0} \cap \text{dcl}(A)^{\perp}$ is infinite and the set $\{b' \in H : \text{tp}(b'/A) = \text{tp}(b/A)\}$ is unbounded, thus $b \notin \text{acl}(A)$. \square

Proposition 3.17 *Let $p, q \in S_1(\emptyset)$ and let $a \models p$, $b \models q$. Then $d(p, q) = \sqrt{\sum_{\lambda \in \sigma} (\|P_{\lambda}(a)\| - \|P_{\lambda}(b)\|)^2}$.*

Proof.

“ \geq ” It is easy to see that

$$\begin{aligned} \|a - b\| &= \sqrt{\|a\|^2 - 2\langle a, b \rangle + \|b\|^2} \\ &= \sqrt{\sum_{\lambda \in \sigma} (\|P_{\lambda}(a)\|^2 - 2\langle P_{\lambda}(a), P_{\lambda}(b) \rangle + \|P_{\lambda}(b)\|^2)} \\ &\geq \sqrt{\sum_{\lambda \in \sigma} (\|P_{\lambda}(a)\|^2 - 2\|P_{\lambda}(a)\| \cdot \|P_{\lambda}(b)\| + \|P_{\lambda}(b)\|^2)} \\ &= \sqrt{\sum_{\lambda \in \sigma} (\|P_{\lambda}(a)\| - \|P_{\lambda}(b)\|)^2}. \end{aligned}$$

Thus

$$d(p, q) = \inf\{\|a' - b'\| : \mathcal{H} \models p(a') \text{ and } \mathcal{H} \models q(b')\} \geq \sqrt{\sum_{\lambda \in \sigma} (\|P_{\lambda}(a)\| - \|P_{\lambda}(b)\|)^2}.$$

“ \leq ” Let a', b' be such that $\text{tp}(a) = \text{tp}(a')$ and $\text{tp}(b) = \text{tp}(b')$. Then

$$\|P_{\lambda}(a)\| = \|P_{\lambda}(a')\| \quad \text{and} \quad \|P_{\lambda}(b)\| = \|P_{\lambda}(b')\|$$

for $\lambda \in \sigma$, but the inner products $\langle P_{\lambda}(a'), P_{\lambda}(b') \rangle$ depend on the choice of a' and b' . We may choose a' and b' such that for each $\lambda \in \sigma$, $\langle P_{\lambda}(a'), P_{\lambda}(b') \rangle = \|P_{\lambda}(a')\| \cdot \|P_{\lambda}(b')\|$. Then

$$\begin{aligned} d(p, q) &= \inf\{\|a - b\| : \mathcal{H} \models p(a) \text{ and } \mathcal{H} \models q(b)\} \\ &\leq \|a' - b'\| \\ &= \sqrt{\|a'\|^2 - 2\langle a', b' \rangle + \|b'\|^2} \\ &= \sqrt{\sum_{\lambda \in \sigma} (\|P_{\lambda}(a')\|^2 - 2\langle P_{\lambda}(a'), P_{\lambda}(b') \rangle + \|P_{\lambda}(b')\|^2)} \\ &= \sqrt{\sum_{\lambda \in \sigma} (\|P_{\lambda}(a')\|^2 - 2\|P_{\lambda}(a')\| \cdot \|P_{\lambda}(b')\| + \|P_{\lambda}(b')\|^2)} \\ &= \sqrt{\sum_{\lambda \in \sigma} (\|P_{\lambda}(a)\| - \|P_{\lambda}(b)\|)^2}. \end{aligned} \quad \square$$

Definition 3.18 Recall that the theory of a metric structure \mathcal{M} is said to be *separably categorical* if whenever $\mathcal{N}_1, \mathcal{N}_2 \models \text{Th}(\mathcal{M})$ are separable we have $\mathcal{N}_1 \cong \mathcal{N}_2$.

Lemma 3.19 *The theory T_σ is separably categorical.*

Proof. Let (\mathcal{H}', U') and (\mathcal{H}'', U'') be separable models of T_σ . Then for each $\lambda \in \sigma$, $\dim(\mathcal{H}'_\lambda) = \dim(\mathcal{H}''_\lambda)$ (which is either finite or \aleph_0). Hence, for every $\lambda \in \sigma$, $\mathcal{H}'_\lambda \cong \mathcal{H}''_\lambda$, and thus $(\mathcal{H}', U') \cong (\mathcal{H}'', U'')$. \square

One could also prove the previous lemma using Proposition 3.17. It is clear that the formula presented in Proposition 3.17 is definable and thus the logic topology and the distance topology agree on the space of types and by [4, Theorem 12.4], T_σ is separably categorical.

Definition 3.20 The theory of a metric structure \mathcal{M} is called ω -stable if for any $\mathcal{N} \models \text{Th}(\mathcal{M})$ and $A \subseteq \mathcal{N}$ countable, $S(A)$ is separable.

Theorem 3.21 *The theory T_σ is ω -stable.*

Proof. Let $\mathcal{H} \models T_\sigma$ be separable and let $A \subseteq H$ be a countable set. Let \bar{A} be the algebraic closure of A and write

$$\mathcal{H} = \bar{A} + \bar{A}^\perp = \bigoplus_{\lambda \in \sigma} \mathcal{H}_\lambda.$$

For $\lambda \in \sigma_e$, let \mathcal{H}'_λ be a separable Hilbert space such that $H_\lambda \cap H'_\lambda = \{0\}$ and $\mathcal{H}' = \mathcal{H} \oplus \bigoplus_{\lambda \in \sigma_e} \mathcal{H}'_\lambda$. We define, for every $v \in H'_\lambda$, $U'_\lambda(v) = \lambda v$, and let U' be the unitary map on \mathcal{H}' induced by U and U'_λ , $\lambda \in \sigma_e$. Then $(\mathcal{H}', U') \models T_\sigma$ and $\dim(H'_\lambda \cap \bar{A}^\perp) = \infty$ for each $\lambda \in \sigma_e$. In particular, by Theorem 3.14, (\mathcal{H}', U') realizes all types over A , so we can work inside the structure (\mathcal{H}', U') . Since (\mathcal{H}', U') is separable, so is $S(A)$. \square

3.2 Countably infinite spectrum

Our next step is to consider a unitary operator U with countable spectrum σ . We write $\sigma = \sigma_a \cup \sigma_i$, where σ_i are the isolated points of the spectrum and σ_a are the non-isolated points of the spectrum.

Theorem 3.22 *The following properties are true:*

1. *The theory T_σ admits quantifier elimination. Furthermore, for any $\bar{a} = (a_1, \dots, a_n) \in H^n$, the type $\text{tp}(\bar{a})$ is determined by the values $\langle P_\lambda(a_k), P_\lambda(a_j) \rangle$ for $\lambda \in \sigma$, $j, k \leq n$.*
2. *T_σ is ω -stable.*

Proof.

1. It follows from the fact that U^* is quantifier-free definable in terms of U that $(U + U^*)/2$ and $(U - U^*)/2$ are quantifier-free definable. Applying Theorem 2.23 for any $v \in H$ and $\lambda \in \sigma$, $P_\lambda(v)$ is in the quantifier-free definable closure of v . The rest of the proof follows as in Theorem 3.14.

2. We can proceed as in Theorem 3.21. \square

Proposition 3.23 *Let $p, q \in S_1(\emptyset)$ and let $a \models p$, $b \models q$. Then $d(p, q) = \sqrt{\sum_{\lambda \in \sigma} (\|P_\lambda(a)\| - \|P_\lambda(b)\|)^2}$.*

Proof. Similar to Proposition 3.17. \square

Theorem 3.24 *The principal types in $S_1(\emptyset)$ are the ones of elements $a \in \mathcal{H}$ with $P_\lambda(a) = 0$ for $\lambda \in \sigma_a$.*

Proof. We can build a structure $(H', 0, +, \langle, \rangle, U')$ which is elementarily equivalent to $(H, 0, +, \langle, \rangle, U)$ such that $H_\lambda = 0$ for all $\lambda \in \sigma_a$. In this structure the types of elements a such that $P_\lambda(a) \neq 0$ for some $\lambda \in \sigma_a$ are omitted.

Conversely, assume that a is such that $P_\lambda(a) = 0$ for all $\lambda \in \sigma_a$ and let $(\mathcal{H}', 0, +, \langle, \rangle, U')$ be a model of T_σ . By Theorem 3.10 the projections P_λ are definable for every isolated $\lambda \in \sigma$ and thus $H'_\lambda \neq 0$. Let $a'_\lambda \in H'_\lambda$ be such that $\|a'_\lambda\| = \|P_\lambda(a)\|$ for each $\lambda \in \sigma_i$ and let $a' = \sum_{\lambda \in \sigma_i} a'_\lambda$. Then $a' \in H'$ and $\text{tp}(a) = \text{tp}(a')$. \square

One could also use Proposition 3.23 to prove the previous theorem.

Corollary 3.25 *The atomic models of T_σ are the models in which the accumulation points of the spectrum are not eigenvalues.*

Corollary 3.26 *The \aleph_0 -saturated models of T_σ are the ones in which the accumulation points of the spectrum are eigenvalues whose eigenspace is infinite-dimensional.*

Corollary 3.27 *For $\lambda \in \sigma_a$, the set H_λ is not definable and the function P_λ is not definable.*

Proof. Assume, in order to get a contradiction, that H_λ is definable. In an \aleph_0 -saturated structure the statement $\inf_{u \in H_\lambda} (\|u\| - 1)^2 = 0$ is true. Thus in the prime model this property also holds and there exists a vector of norm 1 in H_λ , a contradiction. Assume now that P_λ is definable. Then $\|x - P_\lambda(x)\|$ measures the distance from x into $H_\lambda(x)$ and thus H_λ is definable, a contradiction. \square

Theorem 3.28 *If σ_a is finite, then T_σ has \aleph_0 non-isomorphic separable models. If σ_a is infinite, T_σ has 2^{\aleph_0} non-isomorphic separable models.*

Proof. Let \mathcal{H} and \mathcal{H}' be separable models of T_σ . We can write $\mathcal{H} = \bigoplus_{\lambda \in \sigma_i \cup \sigma_a} \mathcal{H}_\lambda$ and $\mathcal{H}' = \bigoplus_{\lambda \in \sigma_i \cup \sigma_a} \mathcal{H}'_\lambda$. For $\lambda \in \sigma_i$, $\mathcal{H}_\lambda \cong \mathcal{H}'_\lambda$, so the only difference between \mathcal{H} and \mathcal{H}' can come from the spaces \mathcal{H}_λ and \mathcal{H}'_λ for $\lambda \in \sigma_a$. For such λ , the spaces \mathcal{H}_λ and \mathcal{H}'_λ are isomorphic if and only if $\dim(\mathcal{H}_\lambda) = \dim(\mathcal{H}'_\lambda)$.

Assume first that σ_a is a finite non-empty set. So there is exactly one model up to isomorphism for every possible dimension of \mathcal{H}_λ for $\lambda \in \sigma_a$. Thus there are \aleph_0 non-isomorphic separable models of the theory T_σ . On the other hand, if σ_a is an infinite countable set, there are 2^{\aleph_0} non-isomorphic separable models of T_σ . \square

3.3 Forking

We fix a countable spectrum σ and a structure $(\mathcal{H}, U) \models T_\sigma$ which is κ -saturated and strongly κ -homogeneous for some uncountable inaccessible cardinal κ . We say $A \subseteq H$ is *small* if $|A| < \kappa$.

Definition 3.29

1. Let $\bar{a} = (a_1, \dots, a_m) \in H^m$, let $B, C \subseteq H$ be small, let $\overline{B \cup C} = \text{acl}(B \cup C)$ and $\bar{C} = \text{acl}(C)$. We say that \bar{a} is **-independent from B over C* and we write $\bar{a} \downarrow_C^* B$ if $P_{\overline{B \cup C}}(P_\lambda(a_i)) = P_{\bar{C}}(P_\lambda(a_i))$ for $i = 1, \dots, m$ and $\lambda \in \sigma$.

2. For $A, B, C \subseteq H$ small we write $A \downarrow_C^* B$ if and only if for all finite subsets \bar{a} of A , we have that $\bar{a} \downarrow_C^* B$.

Lemma 3.30 *Let B be an algebraically closed small set in (\mathcal{H}, U) . Then for every $\lambda \in \sigma$ and for every $a \in H$, $P_\lambda(P_B(a)) = P_B(P_\lambda(a))$.*

Proof. We first need the following claim.

Claim 3.30.1 *P_B and U commute.*

Proof. Let $a \in H$. We can write $a = a_B + a_{B^\perp}$, where $a_B = P_B(a)$ and $a_{B^\perp} = a - a_B$. Then

$$U(a) = U(a_B) + U(a_{B^\perp}).$$

For any $b \in B$, $U(b), U^*(b) \in B$. Thus $\langle b, U(a_{B^\perp}) \rangle = \langle U^*(b), a_{B^\perp} \rangle = 0$. This implies that $U(a_B) \in B$ and $U(a_{B^\perp}) \in B^\perp$. Thus $P_B(U(a)) = P_B(U(a_B) + U(a_{B^\perp})) = U(a_B)$ and thus $P_B(U(a)) = U(P_B(a))$. \square

The result follows from Fact 2.13 part 1. \square

Notation 3.31 For $A \subseteq H$ small, we write \bar{A} for $\text{acl}(A)$.

Corollary 3.32 *Let $\bar{a} = (a_1, \dots, a_m) \in H^m$ and let $B, C \subseteq H$ be small. Then*

$$\bar{a} \downarrow_C^* B \quad \text{if and only if} \quad P_{\overline{B \cup C}}(a_i) = P_{\bar{C}}(a_i) \text{ for } i = 1, \dots, m.$$

Proof. Assume first that $\bar{a} \downarrow_C^* B$, then $P_{\overline{B \cup C}}(P_\lambda(a_i)) = P_{\bar{C}}(P_\lambda(a_i))$ for any $i \leq m$. By Lemma 3.30 we get $P_\lambda(P_{\overline{B \cup C}}(a_i)) = P_\lambda(P_{\bar{C}}(a_i))$ for all λ and thus $P_{\overline{B \cup C}}(a_i) = P_{\bar{C}}(a_i)$.

The converse is proved in a similar way. \square

Proposition 3.33 Let $\bar{a} = (a_1, \dots, a_n) \in H^n$, $\bar{b} = (b_1, \dots, b_m) \in H^m$ and let $C \subseteq H$ be small. Then

$$\bar{a} \downarrow_C^* \bar{b} \text{ if and only if } P_\lambda(a_k) \downarrow_C^* P_\lambda(b_j) \text{ for } k = 1, \dots, n, j = 1, \dots, m, \text{ and } \lambda \in \sigma.$$

Proof. If $\bar{a} \downarrow_C^* \bar{b}$, then $P_{\overline{\{b_1, \dots, b_m\} \cup C}}(P_\lambda(a_i)) = P_{\bar{C}}(P_\lambda(a_i))$ for every $i = 1, \dots, n$ and $\lambda \in \sigma$. For $\lambda \in \sigma$ and $j = 1, \dots, m$ we have that $\bar{C} \subset \bar{C} \cup \{P_\lambda(b_j)\} \subseteq \overline{C \cup \{b_j\}} \subseteq \overline{C \cup \{b_1, \dots, b_m\}}$, so for every $i = 1, \dots, n$, $P_{\overline{\{P_\lambda(b_j)\} \cup C}}(P_\lambda(a_i)) = P_{\bar{C}}(P_\lambda(a_i))$.

Conversely, let $P_{\overline{\{P_\lambda(b_j)\} \cup C}}(P_\lambda(a_k)) = P_{\bar{C}}(P_\lambda(a_k))$ for all $\lambda \in \sigma$, $k = 1, \dots, n$, and $j = 1, \dots, m$. We fix $\lambda \in \sigma$ and write $P_\lambda(a_k) = P_{\bar{C}}(P_\lambda(a_k)) + P_{\bar{C}^\perp}(P_\lambda(a_k))$. Since $P_{\overline{\{P_\lambda(b_j)\} \cup C}}(P_\lambda(a_k)) = P_{\bar{C}}(P_\lambda(a_k))$ for every $j = 1, \dots, m$, then $P_{\bar{C}^\perp}(P_\lambda(a_k)) \perp P_\lambda(b_j)$ and $P_{\bar{C}^\perp}(P_\lambda(a_k)) \perp \overline{C \cup \{P_\lambda(b_j)\}}$ for every $j = 1, \dots, m$. Thus $P_{\bar{C}^\perp}(P_\lambda(a_k)) \perp \overline{C \cup \{P_\lambda(b_1), \dots, P_\lambda(b_m)\}}$, and $P_{\overline{C \cup \{P_\lambda(b_1), \dots, P_\lambda(b_m)\}}}(P_\lambda(a_k)) = P_{\bar{C}}(P_\lambda(a_k))$. Since $P_{\overline{C \cup \{b_1, \dots, b_m\}}}(P_\lambda(a_k))$ belongs to the closed space \mathcal{H}_λ , we also get

$$P_{\overline{C \cup \{P_\lambda(b_1), \dots, P_\lambda(b_m)\}}}(P_\lambda(a_k)) = P_{\overline{C \cup \{b_1, \dots, b_m\}}}(P_\lambda(a_k)).$$

So $P_{\overline{C \cup \{b_1, \dots, b_m\}}}(P_\lambda(a_i)) = P_{\bar{C}}(P_\lambda(a_i))$ for any $\lambda \in \sigma$ and therefore $\bar{a} \downarrow_C^* \bar{b}$. \square

Lemma 3.34 Let $C \subseteq H$ be algebraically closed and let $\bar{a} = (a_1, \dots, a_n) \in H^n$, $\bar{b} = (b_1, \dots, b_m) \in H^m$. Then

$$\bar{a} \downarrow_C^* \bar{b} \text{ if and only if } P_\lambda(a_k) - P_{\bar{C}}(P_\lambda(a_k)) \perp P_\lambda(b_j) \text{ for } k = 1, \dots, n, j = 1, \dots, m, \text{ and } \lambda \in \sigma.$$

Proof. Given two tuples $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_m)$, by Proposition 3.33, $\bar{a} \downarrow_C^* \bar{b}$ if and only if $P_\lambda(a_k) \downarrow_C^* P_\lambda(b_j)$ for $k = 1, \dots, n, j = 1, \dots, m$, and $\lambda \in \sigma$. This happens if and only if

$$P_{\overline{C \cup \{P_\lambda(b_j)\}}}(P_\lambda(a_k)) = P_{\bar{C}}(P_\lambda(a_k)) \text{ for } k = 1, \dots, n, j = 1, \dots, m, \text{ and } \lambda \in \sigma.$$

Finally, $P_{\overline{C \cup \{P_\lambda(b_j)\}}}(P_\lambda(a_k)) = P_{\bar{C}}(P_\lambda(a_k))$ if and only if $P_\lambda(a_k) - P_{\bar{C}}(P_\lambda(a_k)) \perp P_\lambda(b_j)$. \square

Theorem 3.35 The relation \downarrow^* satisfies the following properties: finite character, local character, transitivity, symmetry, invariance, existence, and stationarity.

Proof. We prove all the properties.

1. Finite character: We show that $\bar{a} \downarrow_C^* B$ if and only if $\bar{a} \downarrow_C^* B_0$ for all finite $B_0 \subseteq B$. First of all, if $\bar{a} \downarrow_C^* B$ and $B' \subseteq B$, then $\bar{a} \downarrow_C^* B'$. If $\bar{a} \not\downarrow_C^* B$, then $P_{\overline{B \cup C}}(a_k) \neq P_{\bar{C}}(a_k)$ for some $k = 1, \dots, n$. Let

$$b = P_{\overline{B \cup C}}(a_k) - P_{\bar{C}}(a_k).$$

Then there exist $b_1, \dots, b_l \in B$, $c_1, \dots, c_m \in C$, $\lambda_1, \dots, \lambda_{l+m} \in \sigma$, and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \mathbb{C}$ such that

$$|b - \sum_{k=1}^l \alpha_k P_{\lambda_k}(b_k) - \sum_{j=1}^m \beta_j P_{\lambda_{l+j}}(c_j)| < \|b\|/2.$$

Let $B_0 = \{b_1, \dots, b_l\}$, then $\bar{a} \not\downarrow_C^* B_0$.

2. Local character: For every a and B there exists a sequence $(d_n : n \in \omega)$ such that $d_n \rightarrow P_{\bar{B}}(a)$ and

$$d_n = \alpha_{1n} P_{\lambda_{1n}}(b_{1n}) + \dots + \alpha_{k_n n} P_{\lambda_{k_n n}}(b_{k_n n}), \quad \text{where } b_{ij} \in B, \lambda_{ij} \in \sigma, \text{ and } \alpha_{ij} \in \mathbb{C}.$$

Let $B_0 = \{b_{ij} : i \leq j, j \in \omega\}$. Then $B_0 \subseteq B$ is countable and $a \downarrow_{B_0}^* B$.

3. Transitivity of independence: Let $A \subseteq B \subseteq C \subset H$ be small and let $\bar{a} = (a_1, \dots, a_n)$. If $\bar{a} \downarrow_A^* C$, then $P_{\bar{C}}(a_k) = P_{\bar{A}}(a_k)$ for $k = 1, \dots, n$. So $P_{\bar{C}}(a_k) = P_{\bar{B}}(a_k) = P_{\bar{A}}(a_k)$ for $k = 1, \dots, n$ and therefore $\bar{a} \downarrow_B^* C$ and $\bar{a} \downarrow_A^* B$. The converse is proved in a similar way.

4. Symmetry: It enough to show that for any tuples $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_m)$ and small sets C , $\bar{a} \downarrow_C^* \bar{b}$ if and only if $\bar{b} \downarrow_C^* \bar{a}$. Let $\lambda \in \sigma$, $k \leq n$, and $j \leq m$, we can write

$$P_\lambda(a_k) = P_\lambda(a_C^k) + P_\lambda(a_{\bar{C}^\perp}^k) \quad \text{and} \quad P_\lambda(b_j) = P_\lambda(b_C^j) + P_\lambda(b_{\bar{C}^\perp}^j),$$

where $a_C^k = P_{\bar{C}}(a_k)$, $a_{\bar{C}^\perp}^k = P_{\bar{C}^\perp}(a_k)$. If $\bar{a} \downarrow_C^* \bar{b}$, then, by Lemma 3.34, $P_\lambda(a_k) - P_\lambda(a_C^k) \perp P_\lambda(b_j)$, and then

$$\langle P_\lambda(a_{\bar{C}^\perp}^k), P_\lambda(b_C^j) + P_\lambda(b_{\bar{C}^\perp}^j) \rangle = 0.$$

So

$$\langle P_\lambda(a_{\bar{C}^\perp}^k), P_\lambda(b_C^j) \rangle + \langle P_\lambda(a_{\bar{C}^\perp}^k), P_\lambda(b_{\bar{C}^\perp}^j) \rangle = 0.$$

By Lemma 3.30, $\langle P_\lambda(a_C^k), P_\lambda(b_C^j) \rangle = 0$, so $\langle P_\lambda(a_{\bar{C}^\perp}^k), P_\lambda(b_C^j) \rangle = 0$. On the other hand, by Lemma 3.30, $\langle P_\lambda(a_C^k), P_\lambda(b_{\bar{C}^\perp}^j) \rangle = 0$, so $\langle P_\lambda(a_k), P_\lambda(b_{\bar{C}^\perp}^j) \rangle = 0$. Thus we have $\langle P_\lambda(a_k), P_\lambda(b_j) - P_\lambda(b_C^j) \rangle = 0$ and therefore $P_\lambda(a_k) \perp P_\lambda(b_j) - P_\lambda(P_{\bar{C}}(b_j))$. By Lemma 3.34, this implies that $\bar{b} \downarrow_C^* \bar{a}$ which completes the proof.

5. Invariance: For every $u, v \in H$ and $f \in \text{Aut}(\mathcal{H})$, $\langle u, v \rangle = \langle f(u), f(v) \rangle$. Let $\bar{a} = (a_1, \dots, a_n) \in H^n$ and $\bar{b} = (b_1, \dots, b_m) \in H^m$. Then

$$\begin{aligned} \bar{a} \downarrow_C^* \bar{b} \quad \text{if and only if} \quad & P_\lambda(a_k) - P_\lambda(P_{\bar{C}}(a_k)) \perp P_\lambda(b_j), \\ & \text{for } k = 1, \dots, n, j = 1, \dots, m, \text{ and } \lambda \in \sigma \\ \text{if and only if} \quad & P_\lambda(f(a_k)) - P_\lambda(P_{f(\bar{C})}(f(a_k))) \perp P_\lambda(f(b_j)), \\ & \text{for } k = 1, \dots, n, j = 1, \dots, m, \text{ and } \lambda \in \sigma. \end{aligned}$$

6. Existence: Let $\bar{a} = (a_1, \dots, a_n) \in H^n$ and let $A \subseteq H$ be small. By quantifier elimination, the type $\text{tp}(\bar{a}/A)$ is determined by $P_{\bar{A}}(P_\lambda(a_k))$ for $\lambda \in \sigma$ and the inner products $\langle P_\lambda(a_k), P_\lambda(a_j) \rangle$ for $k, j \leq n$. Let $\bar{b} \in H^n$ and let $B \supseteq A$ be small. Then $\text{tp}(\bar{b}/B)$ is a free extension of $\text{tp}(\bar{a}/A)$ if and only if $\text{tp}(\bar{b}/A) = \text{tp}(\bar{a}/A)$ and $P_{\bar{B}}(P_\lambda(b_k)) = P_{\bar{A}}(P_\lambda(a_k))$ for all $\lambda \in \sigma$ and $k = 1, \dots, n$. For each $\lambda \in \sigma$, let

$$a_{k\bar{A}^\perp}^\lambda = P_{\bar{A}^\perp}(P_\lambda(a_k)) \quad \text{and} \quad a_{k\bar{A}}^\lambda = P_{\bar{A}}(P_\lambda(a_k)).$$

Since B is small, for each $\lambda \in \sigma_e$, $\dim(\mathcal{H}_\lambda \cap B^\perp) = \infty$, so we can find $d_k^\lambda \in H_\lambda \cap B^\perp$ with $\|a_{k\bar{A}^\perp}^\lambda\| = \|d_k^\lambda\|$. Let $b_k = \sum_{\lambda \in \sigma} (a_{k\bar{A}}^\lambda + d_k^\lambda)$ and let $\bar{b} = (b_1, \dots, b_n)$. Then $\text{tp}(\bar{a}/A) = \text{tp}(\bar{b}/A)$ and $P_{\bar{B}}(P_\lambda(b_k)) = P_{\bar{A}}(P_\lambda(a_k))$ for $k = 1, \dots, n$.

7. Stationarity: Let $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_n)$, $\bar{b}' = (b'_1, \dots, b'_n) \in H^n$ and let $A \subseteq B \subseteq H$ be small. Assume that the types of \bar{b} and \bar{b}' over B are free extensions of $\text{tp}(\bar{a}/A)$. Then $\text{tp}(\bar{b}/A) = \text{tp}(\bar{b}'/A)$ and for every $\lambda \in \sigma$ and $i = 1, \dots, n$,

$$P_{\bar{B}}(P_\lambda(b_i)) = P_{\bar{A}}(P_\lambda(b_i)) = P_{\bar{A}}(P_\lambda(b'_i)) = P_{\bar{B}}(P_\lambda(b'_i)).$$

Thus $\text{tp}(\bar{b}/B) = \text{tp}(\bar{b}'/B)$. Therefore for every $p \in S(A)$, $|\{\text{tp}(\bar{a}/B) : \bar{a} \downarrow_A B, p \subseteq \text{tp}(\bar{a}/B)\}| = 1$. □

Corollary 3.36 *The theory T_σ is stable and *-independence coincides with the notion of independence induced by forking.*

Proof. It follows from [4, Theorem 14.14]. □

Observation 3.37 *We had shown in Theorem 3.22 that T_σ is ω -stable and thus it has prime models over sets. Let $A \subseteq H$ be small. For each $\lambda \in \sigma_i$ such that $\dim(\mathcal{H}_\lambda) = \infty$ and $\dim(P_\lambda(\bar{A})) < \aleph_0$, let \mathcal{H}'_λ be a subspace of \mathcal{H}_λ of dimension \aleph_0 . Then $\overline{A \oplus \bigoplus_{\lambda \in \sigma_i} \mathcal{H}'_\lambda}$ is the prime model over A of T_σ .*

It also follows from ω -stability that for every infinite cardinal μ , T_σ has a μ -saturated model of dimension μ . Indeed, for each $\lambda \in \sigma_e$, let \mathcal{H}'_λ be a subspace of \mathcal{H}_λ of dimension μ . Then $\bigoplus \mathcal{H}'_\lambda \models T_\sigma$ has dimension μ and it is saturated.

Definition 3.38 Let $A \subseteq H$ be small and let $p, q \in S_n(A)$. We say that p is almost orthogonal to q ($p \perp^a q$) if $\bar{a} \perp_A \bar{b}$ for all $\bar{a} \models p$ and $\bar{b} \models q$.

Definition 3.39 Let $A \subseteq H$ and let $p, q \in S_n(A)$. We say that p is orthogonal to q ($p \perp q$) if $p_B \perp^a q_B$ for all $B \supseteq A$, $p_B \supseteq p$ a non-forking extension, and $q_B \supseteq q$ a non-forking extension.

Theorem 3.40 Let $A \subseteq H$ be such that $A = \text{acl}(A)$. Let $p, q \in S_1(A)$, $a \models p$ and $b \models q$, let $a = P_A(a) + a'$ and $b = P_A(b) + b'$, and let $\sigma_p = \{\lambda \in \sigma_e : P_\lambda(a') \neq 0\}$, $\sigma_q = \{\lambda \in \sigma_e : P_\lambda(b') \neq 0\}$. Then $p \perp q$ if and only if $p \perp^a q$ if and only if $\sigma_p \cap \sigma_q = \emptyset$.

Proof. Assume that $p \perp q$. Then $p \perp^a q$. Let $a \models p$ and $b \models q$, and let $a' = a - P_A(a)$ and $b' = b - P_A(b)$. The type $\text{tp}(a/A)$ is determined by $P_A(a)$ and the norms of $P_\lambda(a')$ for $\lambda \in \sigma_e$. Assume, in order to get a contradiction, that $P_{\lambda_0}(a') \neq 0$ and $P_{\lambda_0}(b') \neq 0$ for some $\lambda_0 \in \sigma_e$. Let a'', b'' be such that $\text{tp}(a'/A) = \text{tp}(a''/A)$, $\text{tp}(b'/A) = \text{tp}(b''/A)$, and $P_{\lambda_0}(a'')$ is a multiple of $P_{\lambda_0}(b'')$. Then

$$\text{tp}(P_A(a) + a''/A) = \text{tp}(a/A) \quad \text{and} \quad \text{tp}(P_A(b) + b''/A) = \text{tp}(b/A),$$

but $(P_A(a) + a'') \not\perp_A (P_A(b) + b'')$, a contradiction to $p \perp^a q$.

Conversely, assume that $\sigma_p \cap \sigma_q = \emptyset$. Let $B \supseteq A$ and let $p_B \supseteq p$, $q_B \supseteq q$ be non-forking extensions. Let c and d be realizations of p_B and q_B , respectively. We may write $c = c_{\bar{B}} + c'$, $d = d_{\bar{B}} + d'$, where $c_{\bar{B}}, d_{\bar{B}}$ are the projections of c and d over $\text{acl}(B) = \bar{B}$. Then $\sigma_{p_B} = \sigma_p$ and $\sigma_{q_B} = \sigma_q$. Then

$$\langle c', d' \rangle = \langle \sum_{\lambda \in \sigma_p} P_\lambda(c'), \sum_{\mu \in \sigma_q} P_\mu(d') \rangle = \sum_{\lambda \in \sigma_p} \sum_{\mu \in \sigma_q} \langle P_\lambda(c'), P_\mu(d') \rangle = 0.$$

Thus $c' \perp d'$ and $c \perp_B d$. □

A generalization of the previous result appears in [6] when $\sigma_{\text{fin}} = \emptyset$ and $A = \emptyset$.

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