3rd Jairo Charris Seminar

Symmetries of Differential and Difference Equations

Universidad Sergio Arboleda
July 31th - August 2th, 2009
Bogotá, Colombia

Conference Program

Universidad Sergio Arboleda
Calle 74 no. 14-14
Bogotá, COLOMBIA

Program:

- **Friday 31th, classroom A401**

15h-15h30 - Registration and Opening.

15h30-16h30 - Ibragimov, N. H.,
A Survey of Modern Group Analysis of Linear and Nonlinear Problems.

16h30-17h30 - Jiménez, S.,
Symmetries of PDE systems and differential Correspondences.

17h30-18h - Coffee Break

18h-19h - Vargas, A.,
Killing Spinors, superalgebras and symmetries on Riemannian manifolds.

19h-20h - Mozo Fernández, J.
Monomial summability and linear differential equations.

- **Saturday 1st, classroom A601**

8h-9h - Sauloy, J.,
Analytic classification of q-difference equations.

9h-10h - Wolf, K. B.,
The finite $\mathsf{U}(2)$ *oscillator and all aberrations of finite signals.*

10h-10h30 - Coffee Break

10h30-11h30 - Dueñas, H.,
The Laguerre Sovolev-type Orthogonal polynomials, a non diagonal case.

11h30-12h30 - Weil, J.-A.
A prolongation method to compute expansions of (hidden) first integrals along a solution of a differential system

- **Sunday 2nd, classroom A601**

8h30-9h30 - Olver, P. J.,
Applications of Moving Frames.

9h30-10h30 - Suer, S.,
Automorphism Groups and Model Theory.

10h30-11h - Coffee Break

11h-12h - Ismail, T.,
Mittag-Leffler Type Functions and their Applications.

12h-13h - Gazizov, R. K.,
Approximate symmetries and solutions of differential equations with a small parameter.

The Laguerre Sobolev-type Orthogonal polynomials, a non diagonal case

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Consider the polynomials orthogonal with respect to the following Sobolev-type inner product

$$\langle p, q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \boldsymbol{P}(0)^t A\boldsymbol{Q}(0),\ \alpha > -1,$$

where p and q are polynomials with real coefficients,

$$A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix},\ \boldsymbol{P}(0) = \begin{pmatrix} p(0) \\ p'(0) \end{pmatrix},\ \boldsymbol{Q}(0) = \begin{pmatrix} q(0) \\ q'(0) \end{pmatrix},$$

and A is a positive semidefinite matrix. We study the asymptotic behaviour of the polynomials orthogonal with respect to the previous inner product In particular, we focus our attention on their outer relative asymptotics with respect to the standard Laguerre polynomials as well as on an analog of the Mehler-Heine formula for the rescaled polynomials.On the other hand we find an holonomic equation that these polynomials satisfy.

Approximate symmetries and solutions of differential equations with a small parameter

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The theory of approximate transformation groups, recently developed in [1], [2] (see also [3]) is fruitfully used for investigation of symmetry properties of differential equations with a small parameter. In particular, approximate symmeries provide more possibilities for using group analysis methods in constructing solutions of the equations under consideration [4].

The present talk is a survey of certain results of approximate transformation group theory, algorithms for calculation of approximate symmetries and approximate solutions of equations with a small parameter. New results concerning classifications of approximate transformations groups on a plane and ordinary differential equations with a small parameter are also presented.

[1] A join work with Francisco Marcellán Español, Universidad Carlos III de Madrid

References

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2. **Baikov V.A., Gazizov R.K., and Ibragimov N.H.**, Approximate groups of transformations. — Differentsial'nye Uravneniya, 1993. V. 29, pt. 10, pp. 1712–1732; English transl. in Differential Equations, 29, pt. 10 (1993), pp. 1487–1506.
3. **Baikov, V.A., Gazizov, R.K. and Ibragimov, N. H.**, Approximate transformation groups and deformations of symmetry Lie algebras, Chapter 2 in: *CRC Handbook of Lie Group Analysis. Vol. 3: New Trends in Theoretical Developments and Computational Methods*, ed. N. H. Ibragimov, CRC Press, Boca Raton, 1996.
4. **Bagderina, Yu. Yu. and Gazizov, R.K.**, Invariant representation and symmetry reduction for differential equations with a small parameter. — Communications in Nonlinear Science and Numerical Simulation, 2004. V. 9, number 1, pp. 3–11.

A Survey of Modern Group Analysis of Linear and Nonlinear Problems

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Lie group analysis provides a rigorous mathematical formulation of intuitive ideas of symmetry and invariance and furnishes a universal approach to analytical investigations of linear and nonlinear mathematical models. It is particularly useful for solving nonlinear equations analytically when other means of integration fail.

The present talk is a survey of the basic methods from classical Lie group theory and recent investigations of the author. The methods are applied to Riemannian geometry (local theory) and differential equations. The main topics will be on:

1) Lie's method of integration of nonlinear ordinary differential equations,
2) Linear hyperbolic differential equations in Riemannian spaces with non-trivial conformal group: Huygens' principle and solution of Cauchy's problem,
3) Solution of Laplace's problem on invariants of hyperbolic equations,
4) Extension of Euler's method to parabolic equations,
5) Extension of Euler's equation,
6) Integrating factors for higher-order differential equations.

Mittag-Leffler Type Functions and their Applications

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Mittag-Leffler [Mittag-Leffler, G. (1903), Sur la nouevelle function $E_\alpha(x)$, C.R. Acad. Sci. Paris, (Ser. II), 137, 554-558] introduced a function defined by an infinite series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0$$

and investigated some of its properties. This is an entire function of order $\frac{1}{\alpha}$.
Another function having similar properties to those of Mittag-Leffler functions is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta > 0.$$

For $\beta = 1$, $E_{\alpha,1} = E_\alpha$. Such functions arise naturally in the solution of fractional integral equations [Saxena, R., Mathai, A. and Haubold, H. (2002). On fractional kinetic equations, Astrophysics and Space Science, 282, 281-287] and especially in the study of the fractional kinetic equations, random walks, etc.
We study Mittag-Leffler type functions and derive some of their properties including integrals and recurrence relations. We also study fractional kinetic equations of the form

$$N(t) - N_0 = -c_0 D_t^{-1} N(t)$$

and its generalization, where ${}_0D_1^\nu$ is Riemann-Lioville operator of fractional integration.

Symmetries of PDE systems and differential Correspondences

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A PDE system $\mathcal{R} \subseteq J_m^1 M$ can be prolonged to another one $\mathcal{R}^* \subseteq T^*M$ [S. Jiménez, J. Muñoz, and J. Rodríguez, *Correspondences between jet spaces and PDE systems*, Journal of Lie Theory **15** (2005),197-218.]. In analogy with the higher order symmetries, symmetries of $\mathcal{R}^*$ will be called higher dimensional symmetries of $\mathcal{R}$. For a wide class of PDE systems we will prove that every (infinitesimal or finite) symmetry of $\mathcal{R}$ comes from another one of $\mathcal{R}^*$. Since $\mathcal{R}^*$ does not have internal (infinitesimal) symmetries, this fact allows us, in the infinitesimal case, to compute the internal symmetries of $\mathcal{R}$ as external symmetries of $\mathcal{R}^*$. We will also give an algorithmic method to obtain explicit solutions of $\mathcal{R}$ invariant by a given internal symmetry.

Monomial summability and linear differential equations

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Monomial summability has been introduced by M. Canalis-Durand, R. Schäfke and the author in order to study some class of singularly perturbed differential equations.

In this talk we will focus on linear differential equations, with an irregular singularity and depending on a singular parameter. We will explain how to use monomial summability in order to study the Stokes phenomena for this kind of equations. Another possible applications will be commented, as to the analytic classification of singularities of resonnant foliations in dimension two.

Applications of Moving Frames

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In this talk, I will describe a new approach to the theory of moving frames that is based on equivariant maps. The method is completely algorithmic, and can be readily applied to completely general finite-dimensional Lie group and even infinite-dimensional pseudo-group actions. After introducing the basic ideas, I will attempt to survey a wide variety of new applications, including classification of differential invariants, object recognition in computer vision, invariant variational problems and differential equations, and the design of symmetry-preserving numerical approximations.

Analytic classification of q-difference equations

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The program of analytic classification of q-difference equations was first proposed by Birkhoff in [1] in the context of a unified treatment of the Riemann-Hilbert correspondence for fuchsian differential, difference and q-difference equations. In the first part of the talk, I shall present a modern version of this classification obtained ten years ago [4, 5].

The classification program was extended by Birkhoff and Guenter in [2] for irregular equations, but never pursued. In the second part of the talk, I shall present the solution obtained with

Jean-Pierre Ramis and Changgui Zhang in the last decade [3]. If time permits, I shall discuss the Galois side of the theory.

References

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2. **Birkhoff G.D. and Guenther P.E., 1941.** Note on a Canonical Form for the Linear q-Difference System, *Proc. Nat. Acad. Sci.*, Vol. 27, No. 4, 218–222.
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Automorphism Groups and Model Theory

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Model theory is a branch of mathematical logic which studies *satisfaction* and *definability* in formal languages. It provides a very general framework for doing algebra and geometry. Under certain assumptions, model theoretic machinery can be used to introduce invariants like rank and dimension. Also, model theory has a notion called prime model which is a generalization of the algebraic closure of a field, and offers tools to investigate the existence and uniqueness of prime models.

One of the most important tools model theory has developed is Boris Zil'ber's binding group construction which, under a technical assumption called total transcendence, says the following.

Theorem. Let T be a totally transcendental theory and let X and Y be parameter-free definable sets in the prime model M of T over the emptyset. Assume that X is parameter-free internal to Y. Then the elements of $\mathrm{Aut}(M/Y)$ form a definable group of permutations of X.

Here, the assumption about internality can be seen as an abstract variant of existence of a superposition law. In precise terms, a definable set X is internal to another definable set Y, if there is a definable subset Z of some Y^n and a definable surjection from Z onto X. For instance, the solution set of an ordinary linear differential equation, being an n dimensional vector space over the constants, is internal to constants.

Bruno Poizat, in [1], proved that Kolchin's strongly normal theory is a special case of this construction where the role of Y in the theorem above is played by the constants. Moreover, he observed that the same construction should work for any definable set Y, not just for the constants. Later, in [2], Anand Pillay carried out the details of this observation and developed a differential Galois theory in which the groups are differential algebraic groups. In this talk, we will give a survey of Pillay's theory.

References

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Killing Spinors, superalgebras and symmetries on Riemannian manifolds.

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It is well known that Killing vector fields on Riemannian manifolds generate flows of continuous isometries of the underlying space and so they form the Lie algebra of the isometry group of the manifold. In the case the manifold admits spinor fields, there is a related notion of (geometric) Killling spinors which give rise to an associated superalgebra that can be a Lie algebra (or superalgebra) if the Jacobi identity is satisfied. Interestingly, this superalgebras on low dimensional spheres have recently been shown to be the Lie algebras of some exceptional Lie groups.

A prolongation method to compute expansions of (hidden) first integrals along a solution of a differential system

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Consider a differential system $\dot{x} = X(x)$. We look for criteria to prove or disprove the existence of first integrals in some classe of functions, generally meromorphic ones. The celebrated methods of Ziglin and then Morales and Ramis proceed as follows : pick a "nice" particular solution of the above system, compute the linearization of the system along this trajectory (the "variational equation") and build necessary conditions by the latter.
In this work, we will show how expansions of first integrals along a given solutions can be computed. One ingredient (not surprising) is to solve successive variational equations ; another ingredient (more surprising) is the appearance of linear (algebraic) conditions which stem from prolongation conditions for the jets. We will show examples of the power of this method -

[2]This is a joint work with A. Aparicio-Monforte and S. Simón Estrada

which comes both as a partial effective version and a complement to the Higher Variational Techniques developped by Morales, Ramis and Simo.
If the differential system is hamiltonian, one may perform further simplifications. For example, using previous work of Aparicio Monforte and Weil, we may reduce the variational equation to its "sparsest" form and this in turn makes higher variational equations sparse. Examples (and limits) of this will be also given.

The finite $\mathsf{U}(2)$ oscillator and *all* aberrations of finite signals

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The Lie algebra and group $\mathsf{U}(2)$ is the framework for finite Hamiltonian systems with the geometry and dynamics of the harmonic oscillator: $[H, X] = -\mathrm{i}P$ and $[H, P] = \mathrm{i}X$ respectively, for position X, momentum P, and Hamiltonian H. The difference between the finite and the ordinary quantum model lies in the third commutator: $[X, P] = \mathrm{i}H$ (plus constant). Within a definite irreducible representation of dimension $N = 2j + 1$ ($j \in \frac{1}{2}\mathcal{Z}$) we interpret the wavefunctions as N-point signals, such as those produced by an array of light-emitting devices that undergo parallel processing by an optical setup, and are registered on a similar array of sensors. The energy eigenfunctions of the finite oscillator comply with a second-order difference equation —well known by group theorists— that is satisfied by Kravchuk functions; the $N \to \infty$ limit returns the ordinary Hermite-Gauss functions on the continuum. The $\mathsf{U}(2)$ *Fourier-Kravchuk* transforms become the fractional Fourier transforms on $\Re$ in this limit.

The coadjoint orbits of the algebra form an $\Re^3$ space, and the restriction to a definite j reduces this manifold to a sphere, which thus serves as phase space on which to picture finite signals through the definition of a good (covariant, with marginals, etc.) Wigner function.

As in geometric optics, N-point signals in nano-optics generally aberrate, with $\mathsf{U}(N)$ transformations beyond $\mathsf{U}(2)$. We classify their N^2 parameters in correspondence with the geometric nomenclature of paraxial and metaxial transformations, and within the latter, their classification by aberration order and type (spherical aberration, coma, astigmatism, distorsion, …), and show their action on the phase space sphere.

The model to be presented during the Seminar has been used to produce various signal processing algorithms, such as unitary rotation of 2D images and unitary transformations from cartesian to polar pixellations; it has been generalized to other Lie algebras ($\mathsf{I}_2\mathsf{SU}(1)$ for free systems, $\mathsf{U}(1,1)$ for repulsive oscillators, $\mathsf{O}(4)$ for 2D oscillators), and to a finite q-oscillator, where the spectrum of positions is not equally-spaced. These topics will not be included in the talk but conversations on them are welcome.